Variational Inference for Structured NLP Models

ACL, August 4, 2013

David Burkett and Dan Klein

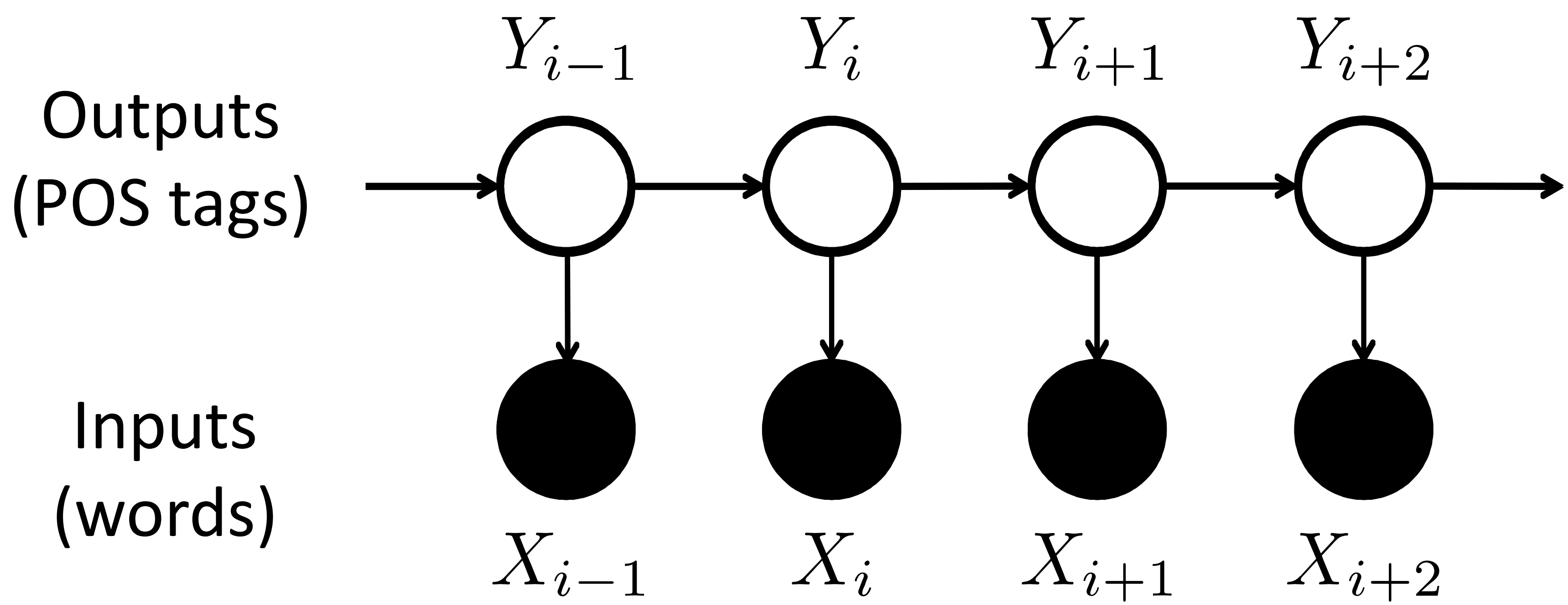
Tutorial Outline

1. Structured Models and Factor Graphs
2. Mean Field
3. Structured Mean Field
4. Belief Propagation
5. Structured Belief Propagation
6. Wrap-Up

Part 1: Structured Models and Factor Graphs

Structured NLP Models

Example: Hidden Markov Model
(Sample Application: Part of Speech Tagging)

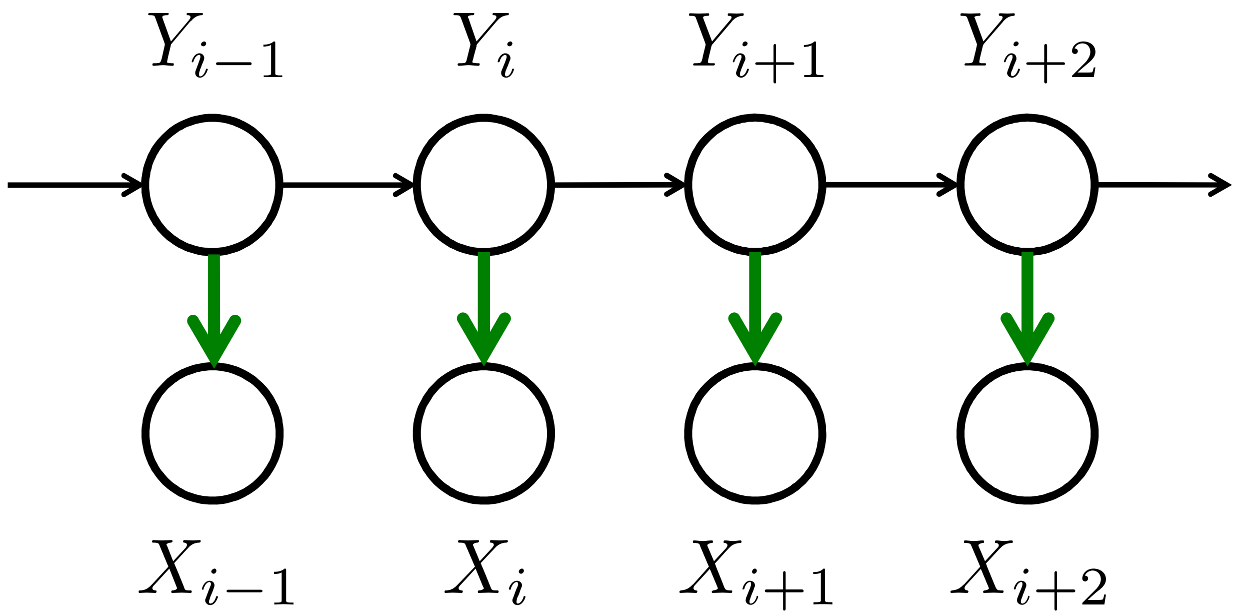


Goal: Queries from posterior $P(Y = y | X = x)$ ($P(y|x)$)

Structured NLP Models

Example: Hidden Markov Model

$$P(x_i|y_i)$$

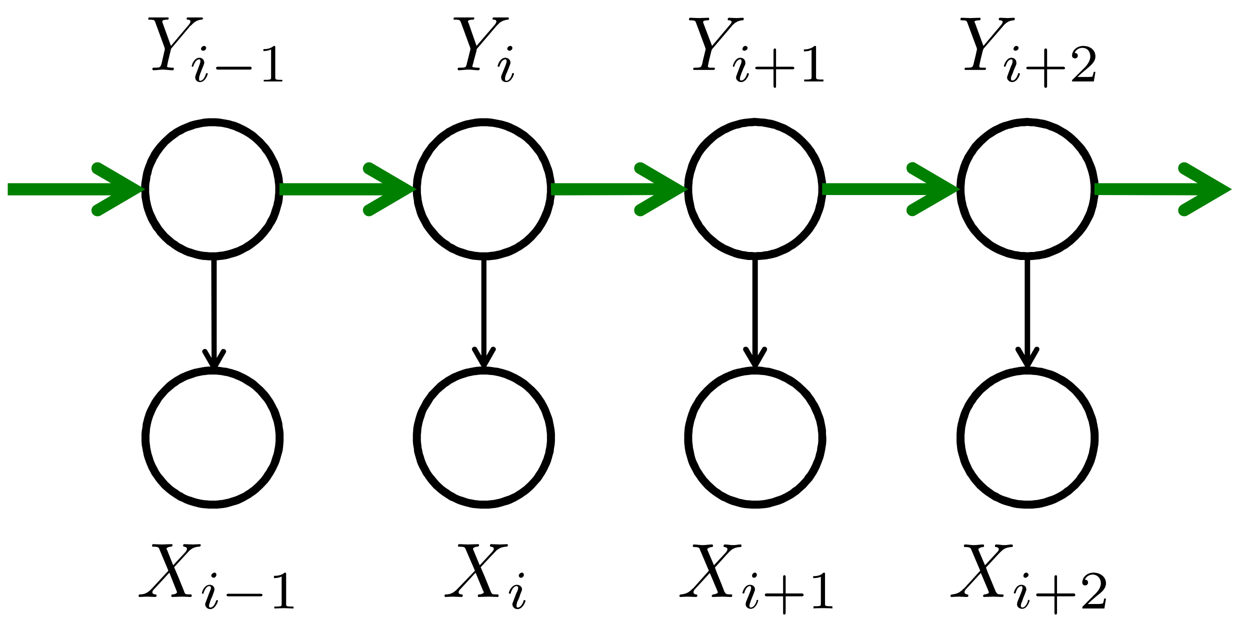


Structured NLP Models

Example: Hidden Markov Model

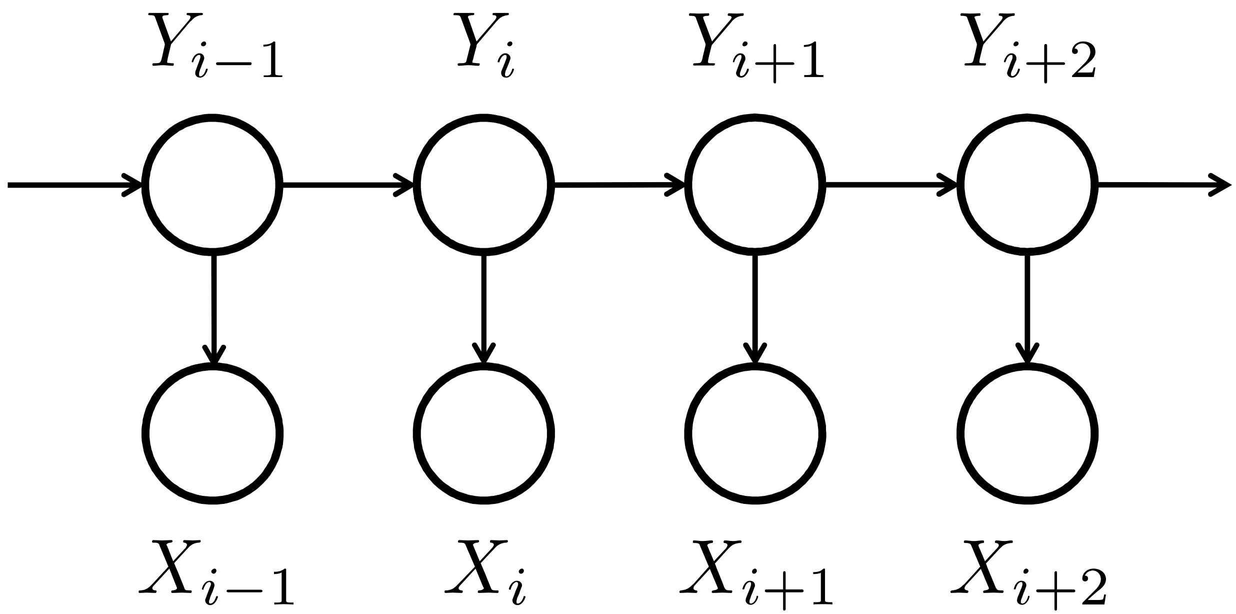
$P(y_{i-1}|y_i)$

$P(x_i|y_i)$



Structured NLP Models

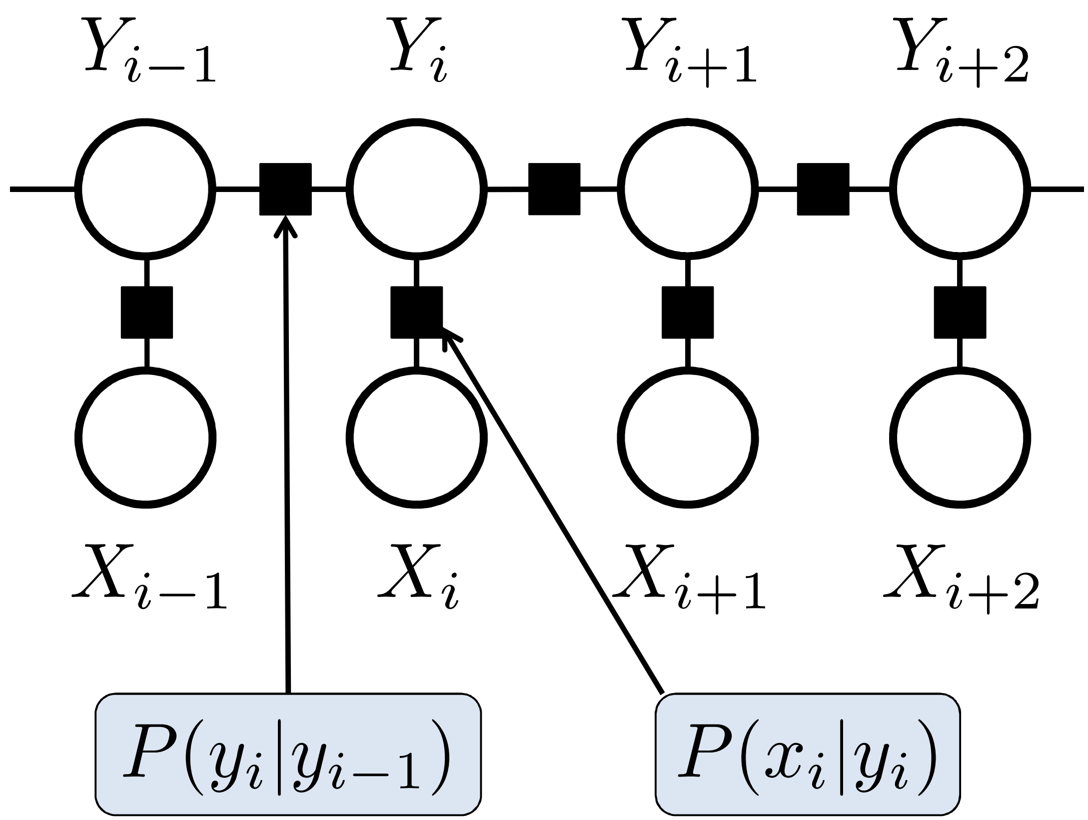
Example: Hidden Markov Model



$$P(y|x) \propto \prod_i P(y_i|y_{i-1})P(x_i|y_i)$$

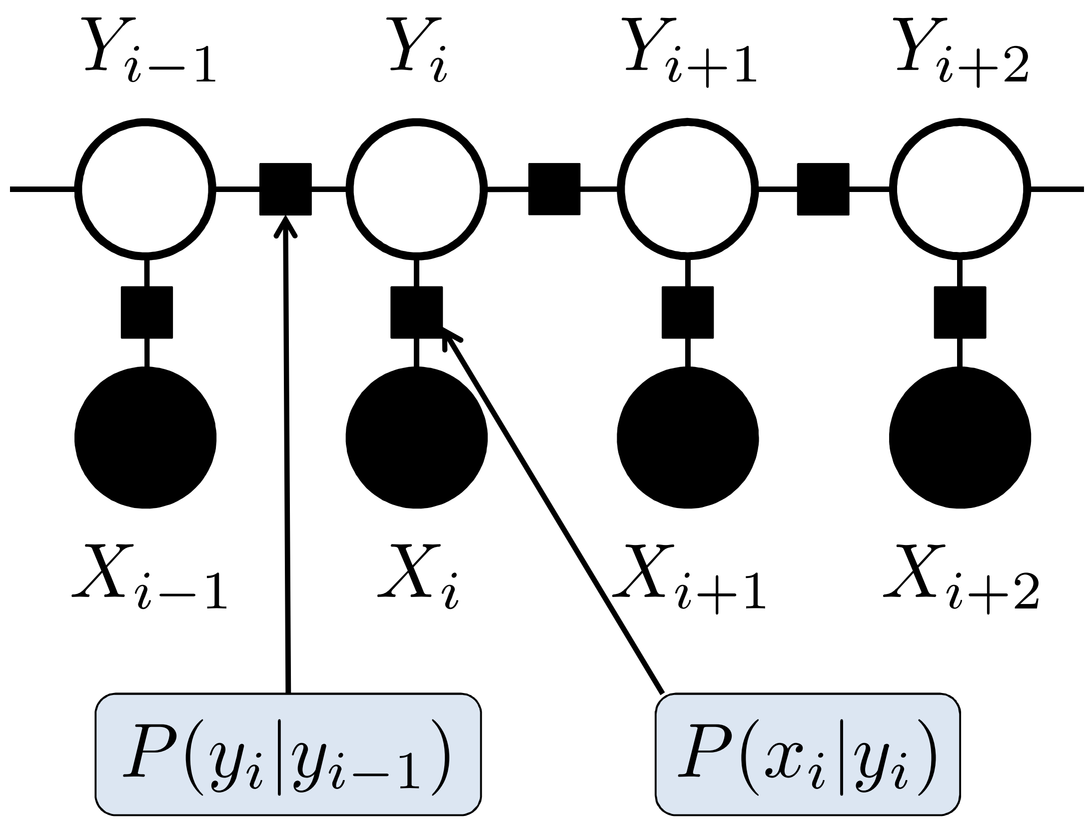
Structured NLP Models

Example: Hidden Markov Model



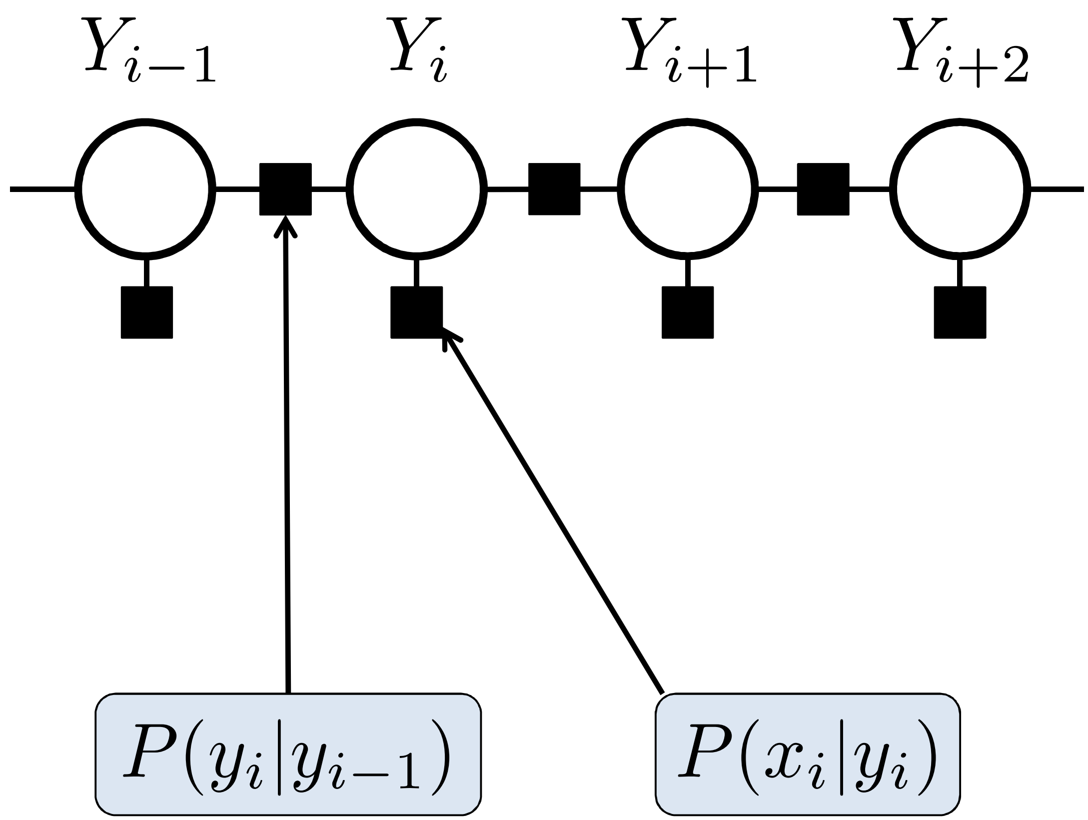
Structured NLP Models

Example: Hidden Markov Model



Structured NLP Models

Example: Hidden Markov Model

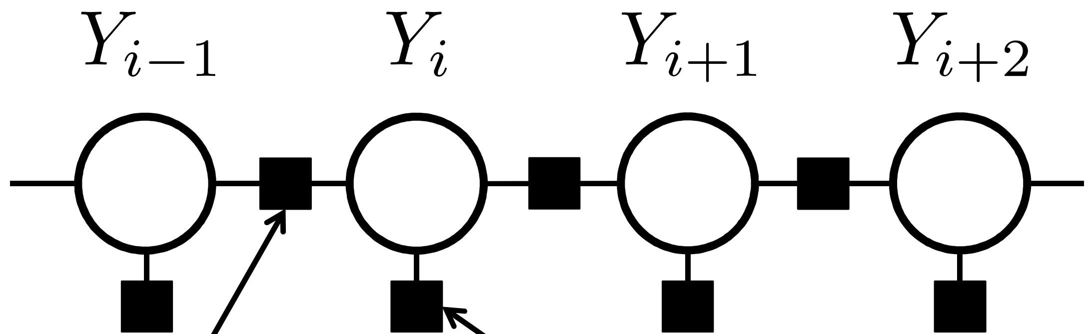


Factor Graph Notation

Variables Y_i

Factors ϕ

Cliques c



Binary Factor

$$\phi_c(y_{i-1}, y_i) = P(y_i | y_{i-1})$$

$$c = \{i-1, i\}$$

Unary Factor

$$\phi_c(y_i) = P(x_i | y_i)$$

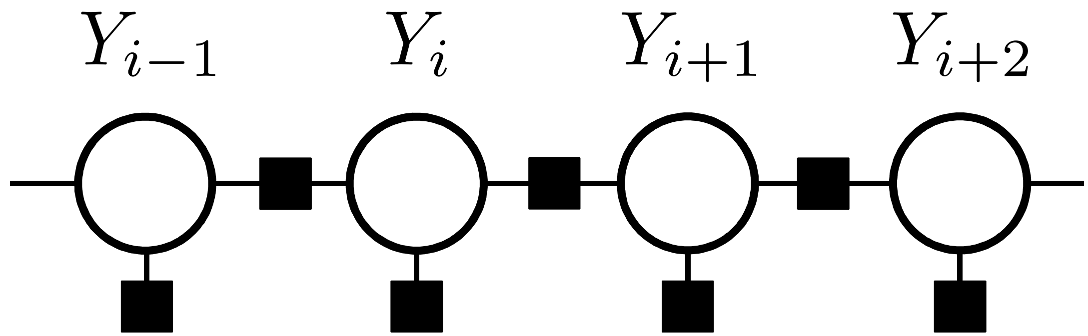
$$c = \{i\}$$

Factor Graph Notation

Variables Y_i

Factors ϕ

Cliques c



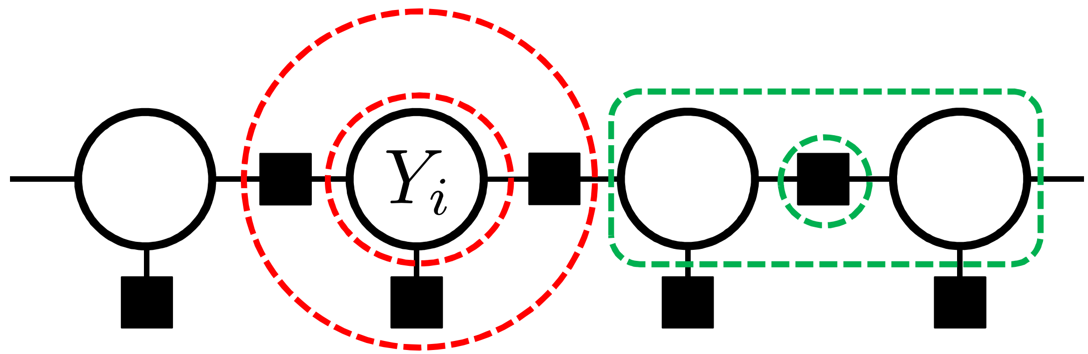
$$P(y|x) \propto \prod_c \phi_c(y_c) = \prod_i P(y_i|y_{i-1})P(x_i|y_i)$$

Factor Graph Notation

Variables Y_i

Factors ϕ

Cliques c



Variables have factor (clique) neighbors:

$$\mathcal{N}(i) = \{c : i \in c\}$$

Factors have variable neighbors:

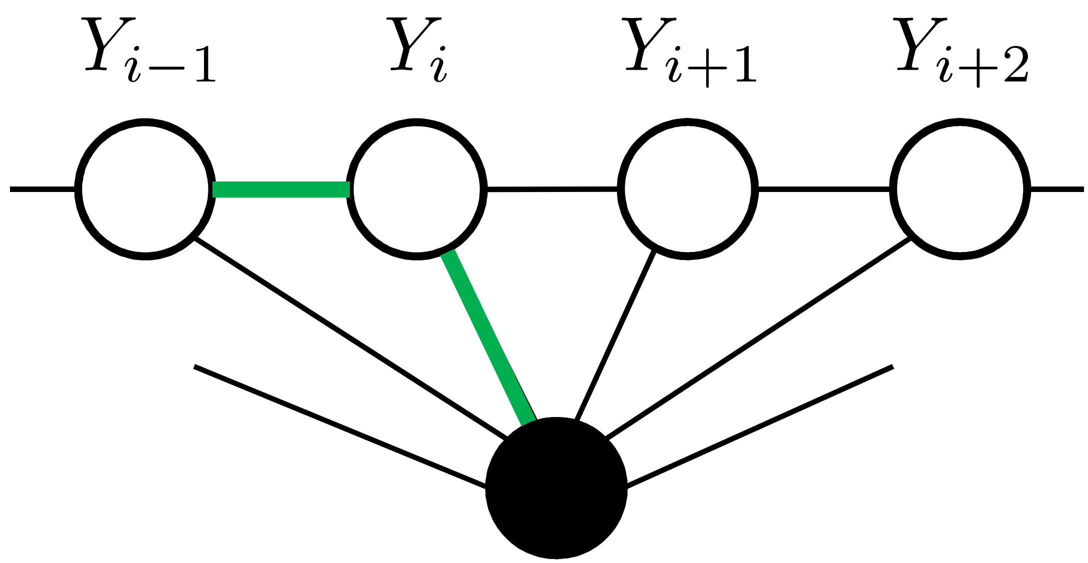
$$\mathcal{N}(\phi_c) = c$$

(Lafferty et al., 2001)

Structured NLP Models

Example: Conditional Random Field

(Sample Application: Named Entity Recognition)

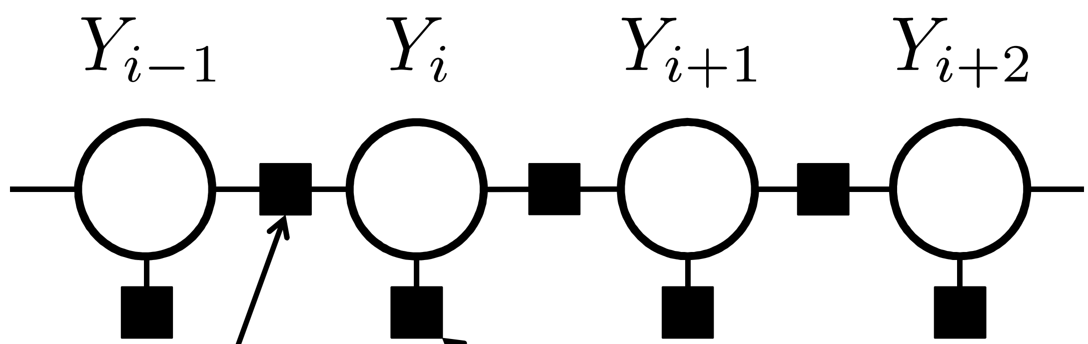


$$X = X_1, \ldots, X_i, \ldots, X_n$$

$$P(y|x) \propto \exp\left(\sum_i w^\top f_i(y_i, x) + w^\top f_i(y_{i-1}, y_i, x)\right)$$

Structured NLP Models

Example: Conditional Random Field



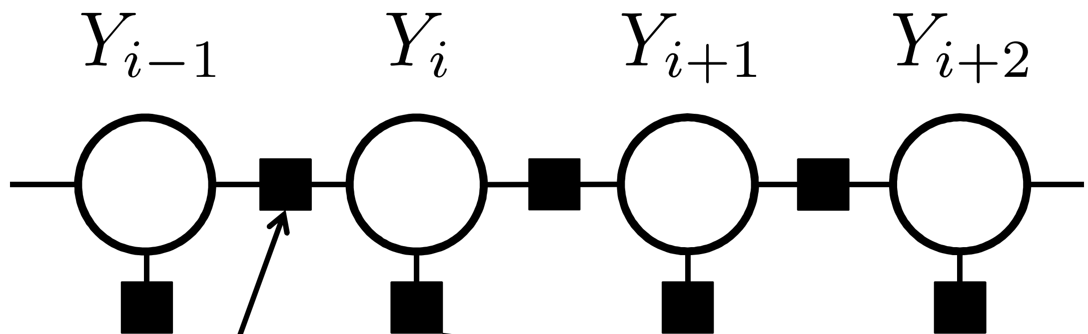
$$\phi_i(y_i) = \exp\left(w^\top f_i(y_i, x)\right)$$

$$\phi_{i-1,i}(y_{i-1}, y_i) = \exp\left(w^\top f_i(y_{i-1}, y_i, x)\right)$$

$$P(y|x) \propto \exp\left(\sum_i w^\top f_i(y_i, x) + w^\top f_i(y_{i-1}, y_i, x)\right)$$

Structured NLP Models

Example: Conditional Random Field



$$\phi(y_i) = \exp\left(w^\top f(y_i)\right)$$

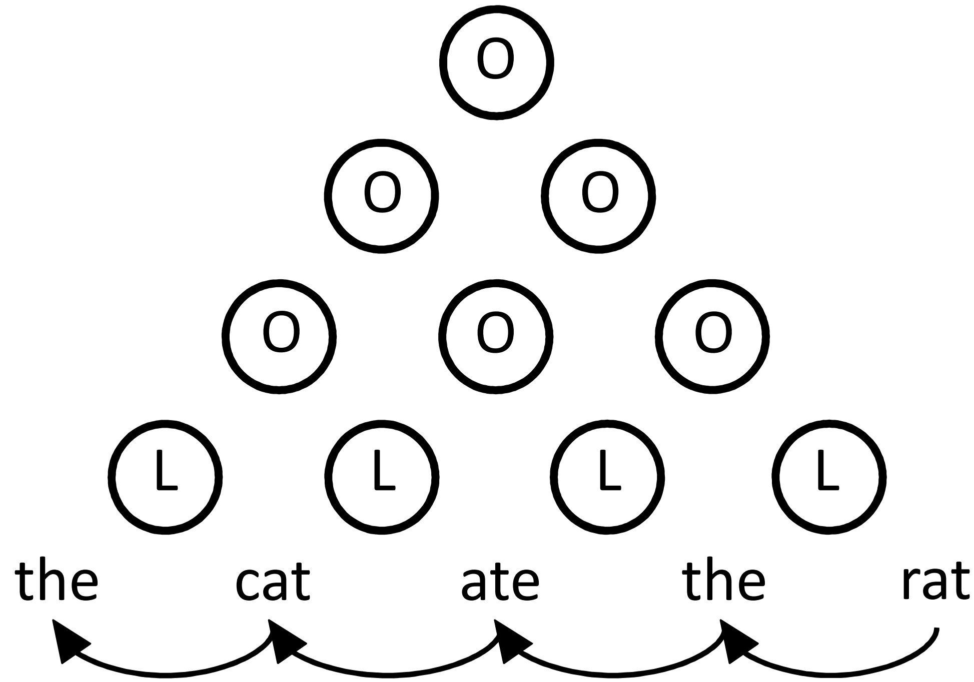
$$\phi(y_{i-1}, y_i) = \exp\left(w^\top f(y_{i-1}, y_i)\right)$$

$$P(y|x) \propto \exp\left(\sum_i w^\top f(y_i) + w^\top f(y_{i-1}, y_i)\right)$$

Structured NLP Models

Example: Edge-Factored Dependency Parsing

$$y_{ij} \in \{\text{left}, \text{right}, \text{off}\}$$



(McDonald et al., 2005)

Structured NLP Models

Example: Edge-Factored Dependency Parsing

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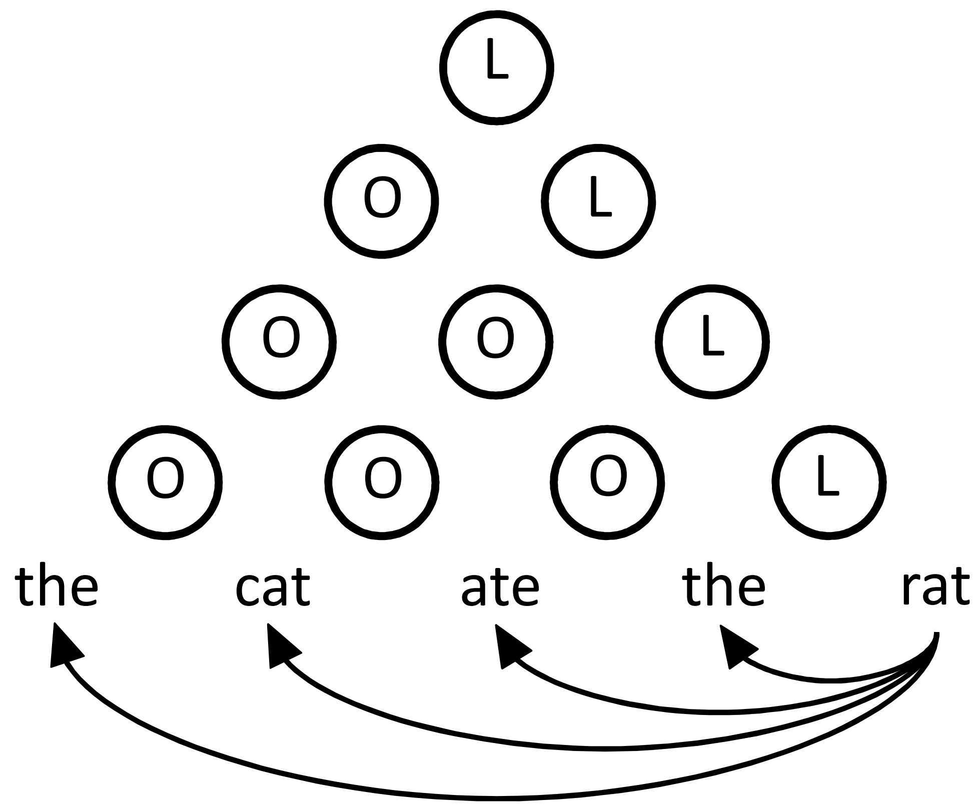

Structured NLP Models

Example: Edge-Factored Dependency Parsing

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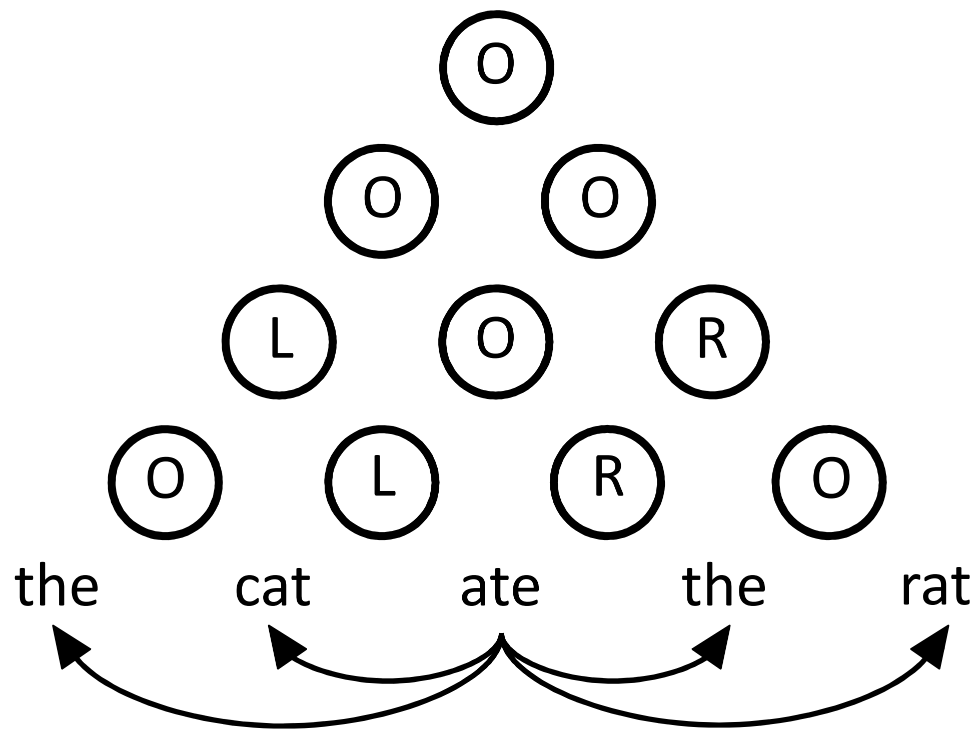

Structured NLP Models

Example: Edge-Factored Dependency Parsing

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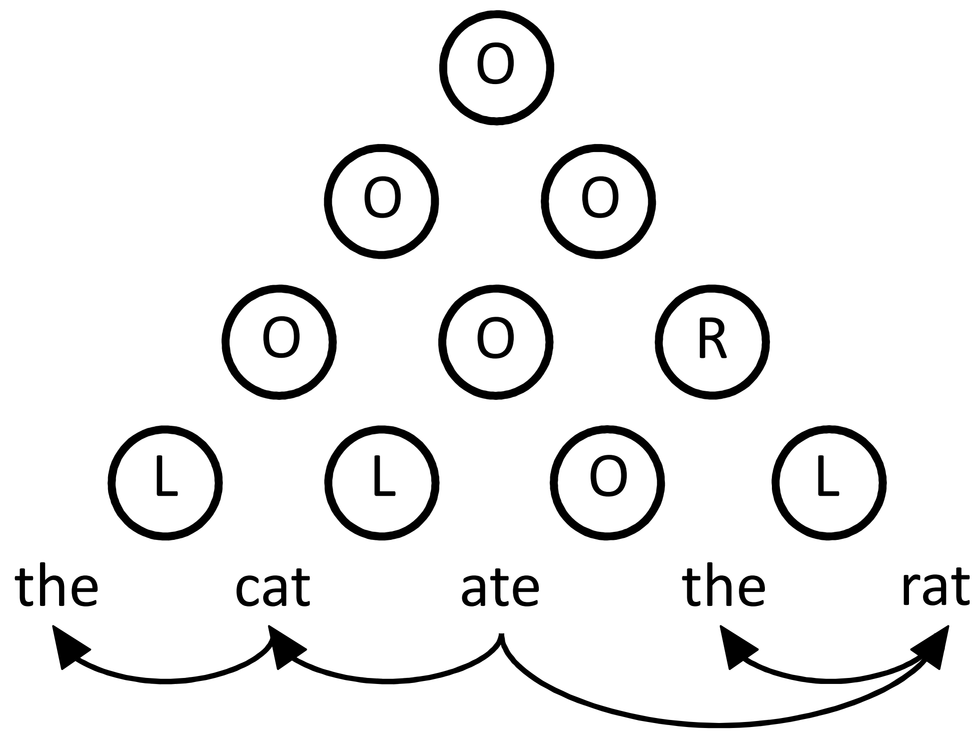
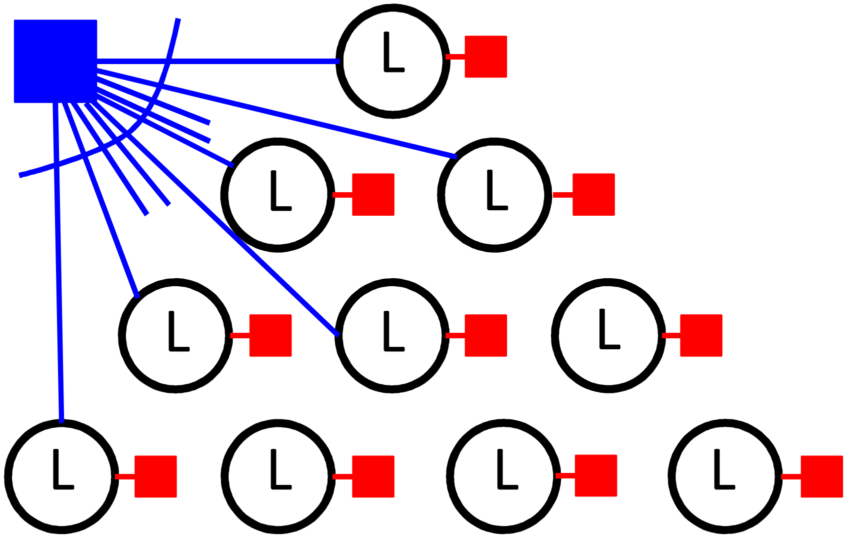


Structured NLP Models

Example: Edge-Factored Dependency Parsing

$$y_{ij} \in \{\text{left, right, off}\}$$

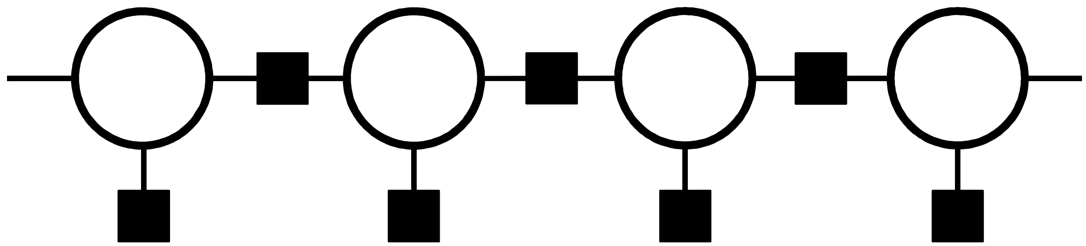


$$\phi(y) = \begin{cases} 1 & y \text{ forms a tree} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(y_{ij}) = \begin{cases} \exp(w^\top f(i,j)) & y_{ij} = \text{left} \\ \exp(w^\top f(j,i)) & y_{ij} = \text{right} \\ 1 & y_{ij} = \text{off} \end{cases}$$

Inference

- Input: Factor Graph



- Output: Marginals $P(y_i|x)$

Inference

- Typical NLP Approach: Dynamic Programs!

- Examples:
  - Sequence Models (Forward/Backward)
  - Phrase Structure Parsing (CKY, Inside/Outside)
  - Dependency Parsing (Eisner algorithm)
  - ITG Parsing (Bitext Inside/Outside)

Complex Structured Models

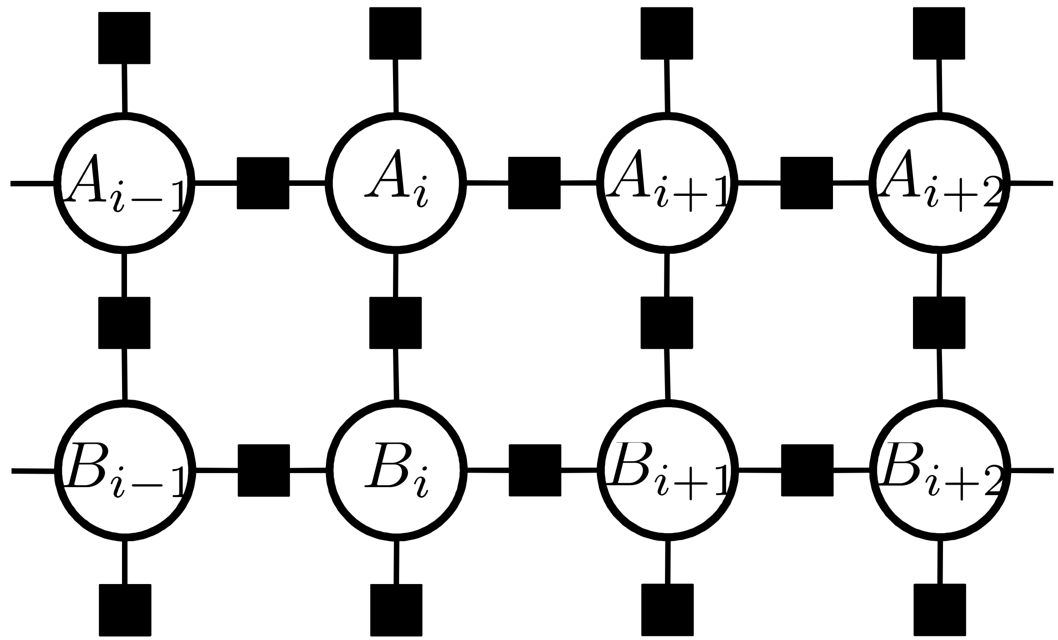
POS Tagging

↑

Joint

↓

Named Entity Recognition

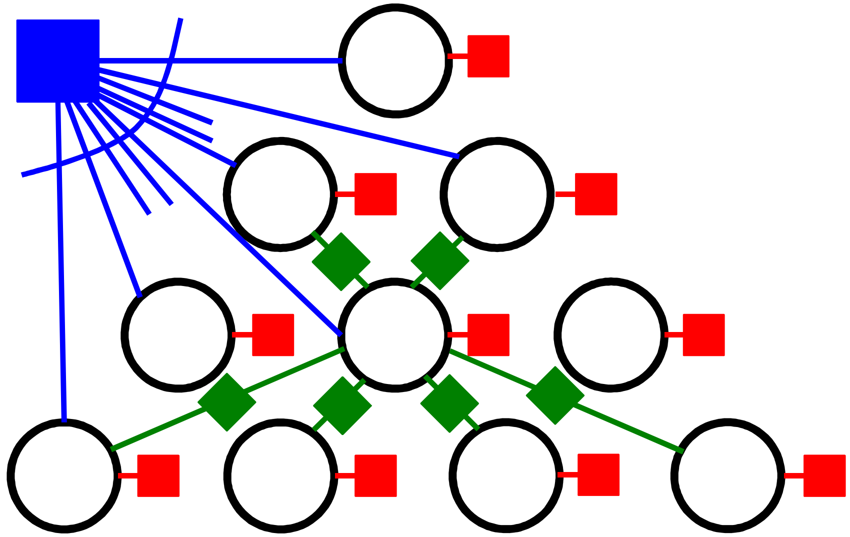


(Sutton et al., 2004)

Complex Structured Models

Dependency Parsing

with Second Order Features



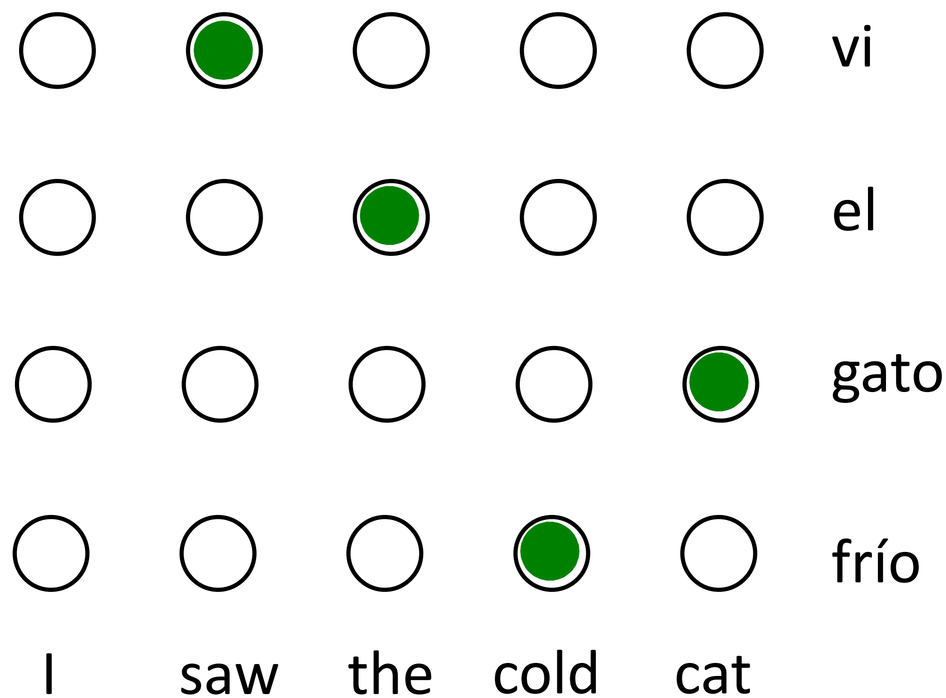
(McDonald & Pereira, 2006)

(Carreras, 2007)

Complex Structured Models

Word Alignment

$y_{ij} \in \{\text{on}, \text{off}\}$



(Taskar et al., 2005)

Complex Structured Models

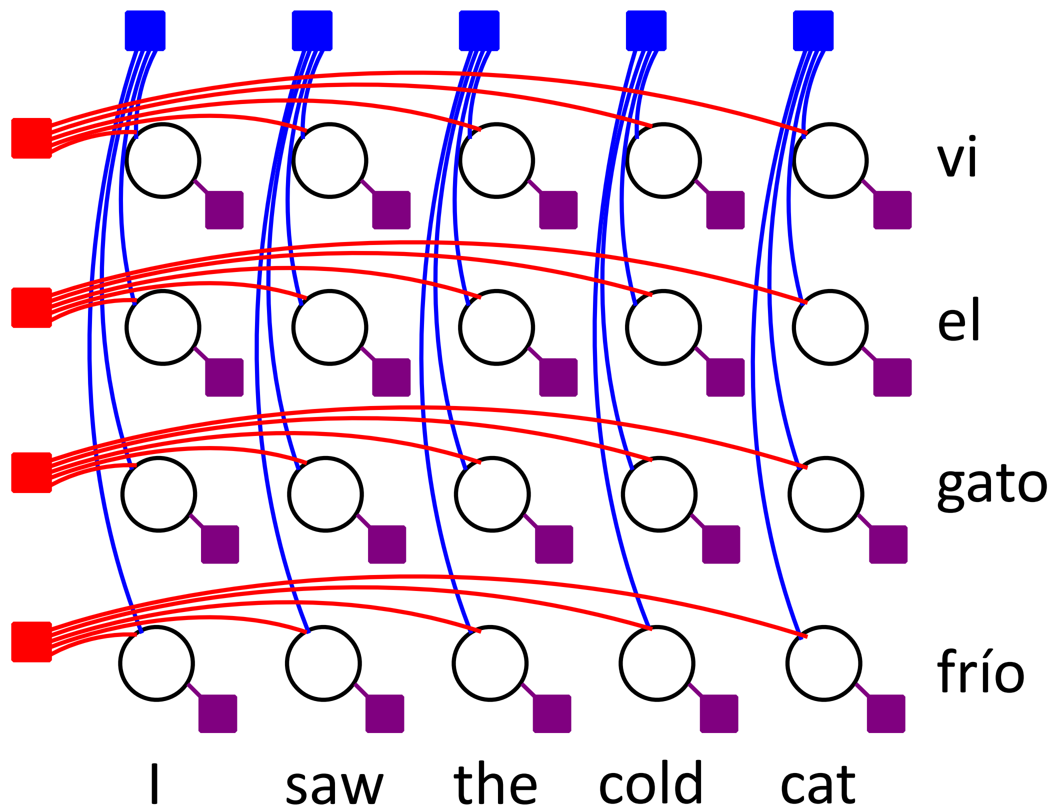
Word Alignment

$$y_{ij} \in \{\text{on}, \text{off}\}$$

$$\phi(y_{ij}) = \begin{cases} \exp(w^\top f(i,j)) & y_{ij} = \text{on} \\ 1 & y_{ij} = \text{off} \end{cases}$$

$$\phi(y_{i*}) = \begin{cases} 1 & |\{j : y_{ij} = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(y_{*j}) = \begin{cases} 1 & |\{i : y_{ij} = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



Variational Inference

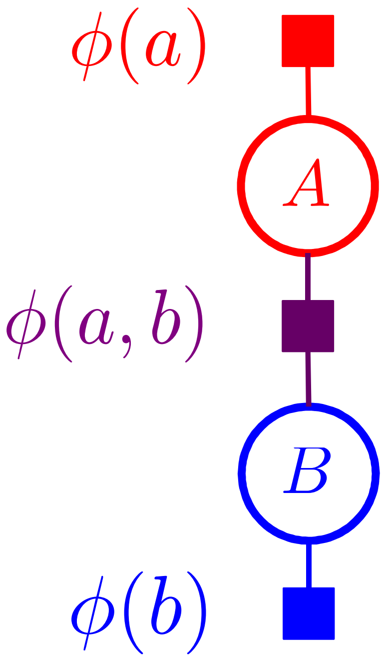
- Approximate inference techniques that can be applied to any graphical model
- This tutorial:
  - Mean Field: Approximate the joint distribution with a product of marginals
  - Belief Propagation: Apply tree inference algorithms even if your graph isn't a tree
  - Structure: What changes when your factor graph has tractable substructures

Part 2: Mean Field

Berkeley

N L P

Mean Field Warmup



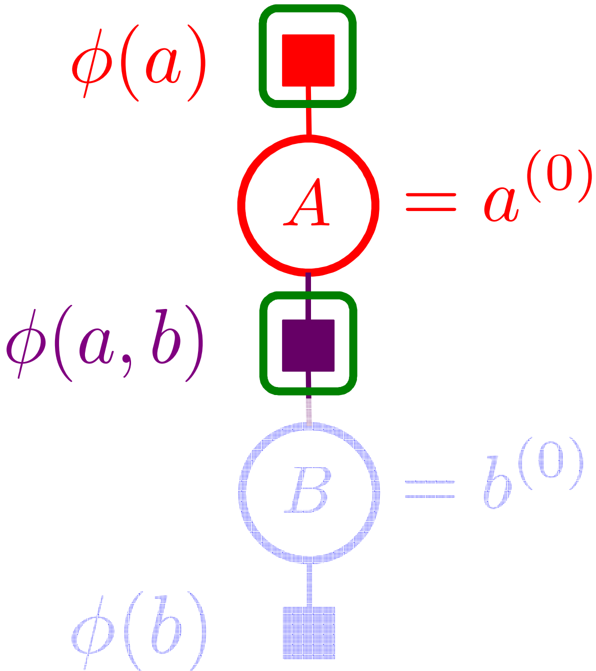
Wanted: $\operatorname*{argmax}_{a,b} P(a, b|x)$

Idea: coordinate ascent

Key object: assignments

Iterated Conditional Modes (Besag, 1986)

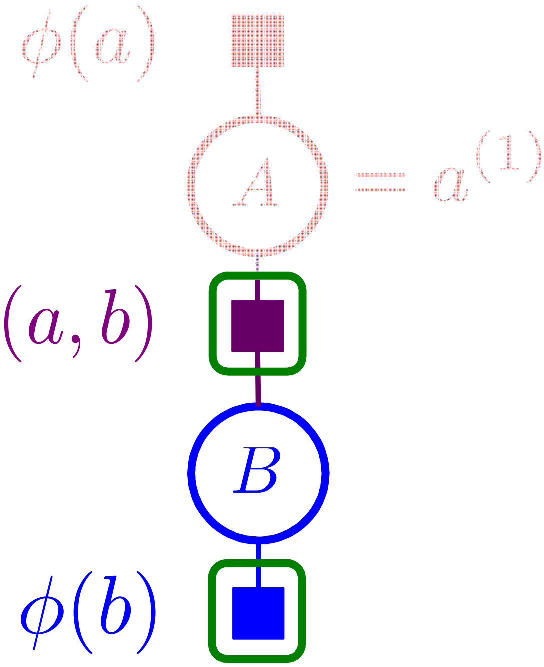
Mean Field Warmup



$$a^{(1)} = \operatorname*{argmax}_{a} \phi(a)\phi(a,b)$$

Wanted: $\operatorname*{argmax}_{a,b} P(a,b|x)$

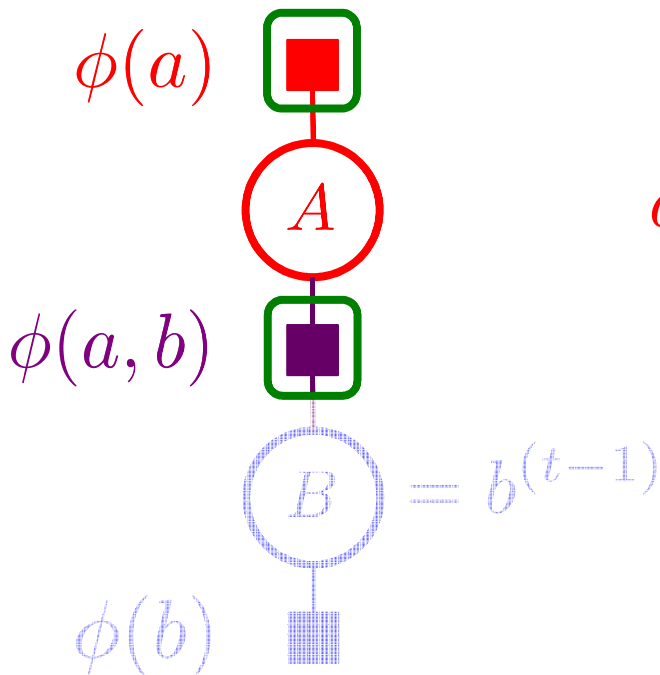
Mean Field Warmup



$$b^{(1)} = \operatorname*{argmax}_{b} \phi(b)\phi(a,b)$$

Wanted: $\operatorname*{argmax}_{a,b} P(a,b|x)$

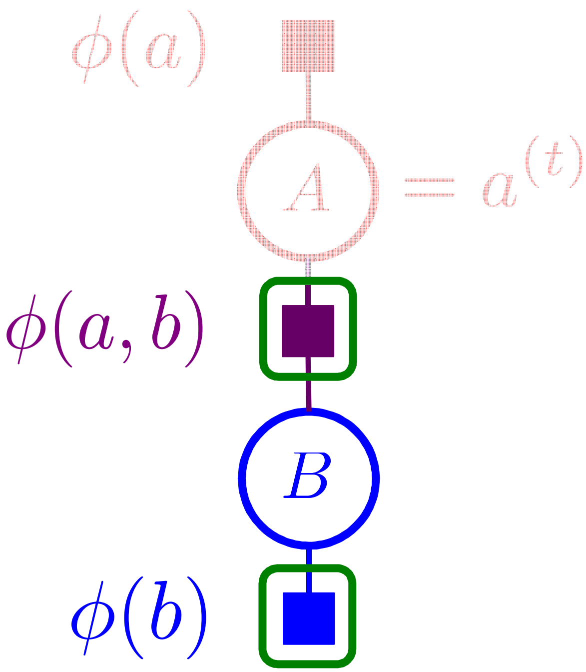
Mean Field Warmup



$$a^{(t)} = \operatorname*{argmax}_{a} \phi(a)\phi(a, b)$$

Wanted: $\operatorname*{argmax}_{a,b} P(a, b|x)$

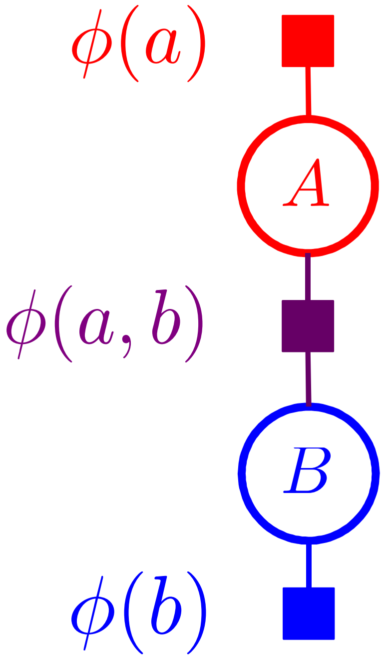
Mean Field Warmup



$$b^{(t)} = \operatorname*{argmax}_{b} \phi(b)\phi(a, b)$$

Wanted: $\operatorname*{argmax}_{a,b} P(a, b|x)$

Mean Field Warmup



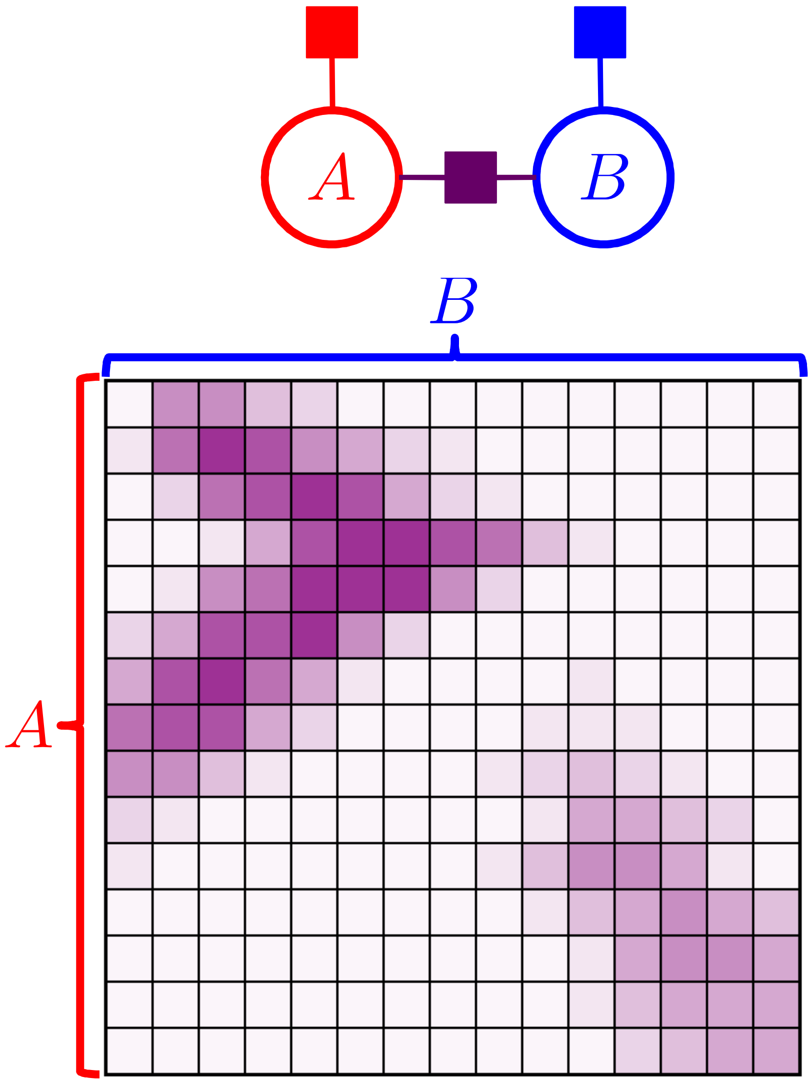
$$a^{(t)} = a^{(t-1)}$$

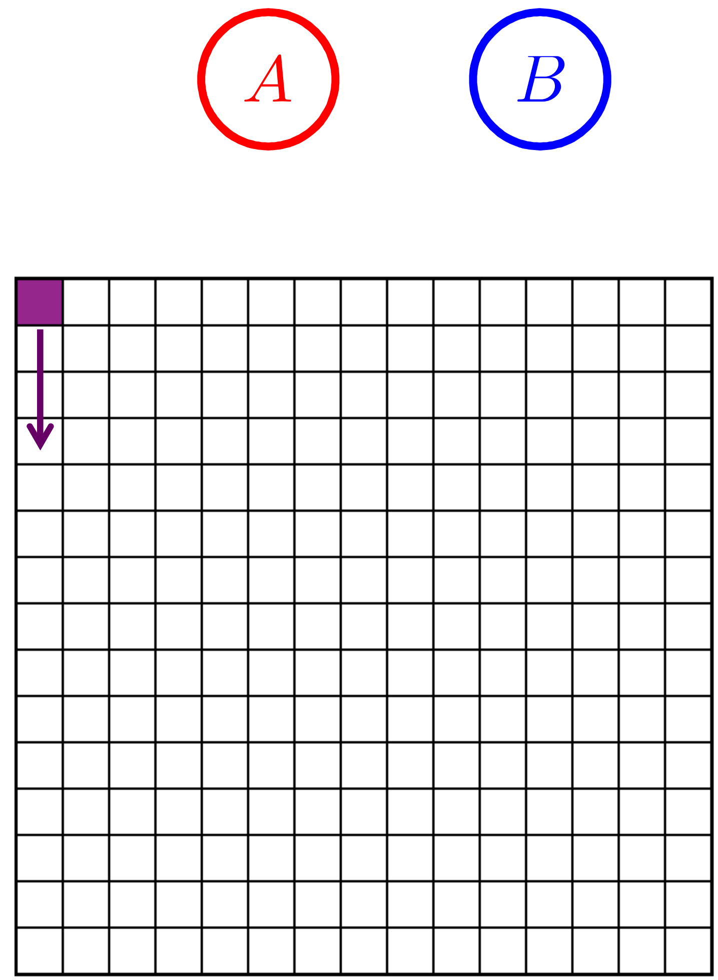
$$b^{(t)} = b^{(t-1)}$$

Wanted: $\underset{a,b}{\operatorname{argmax}} P(a, b|x)$

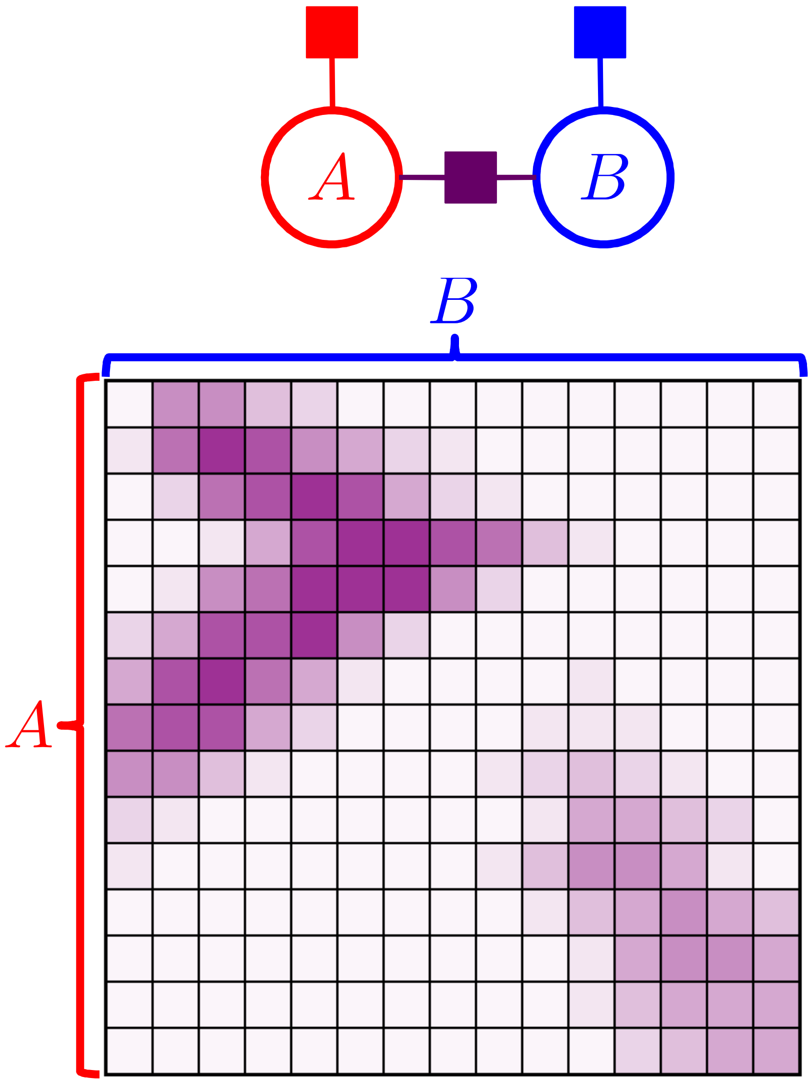
Approximate Result: $(a^{(t)}, b^{(t)})$

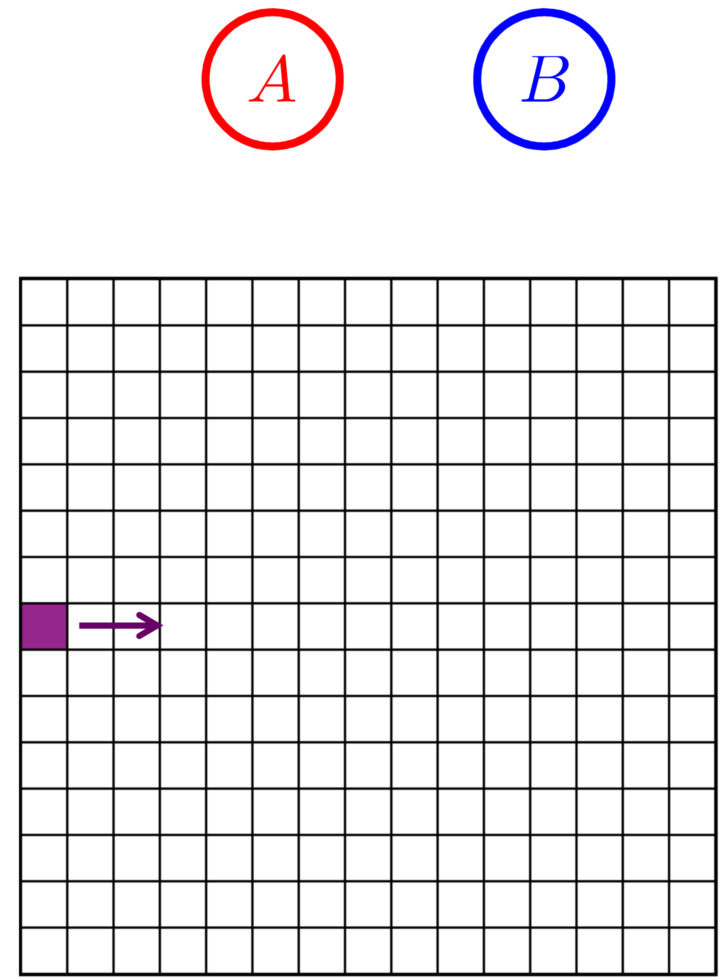
Iterated Conditional Modes Example





Iterated Conditional Modes Example





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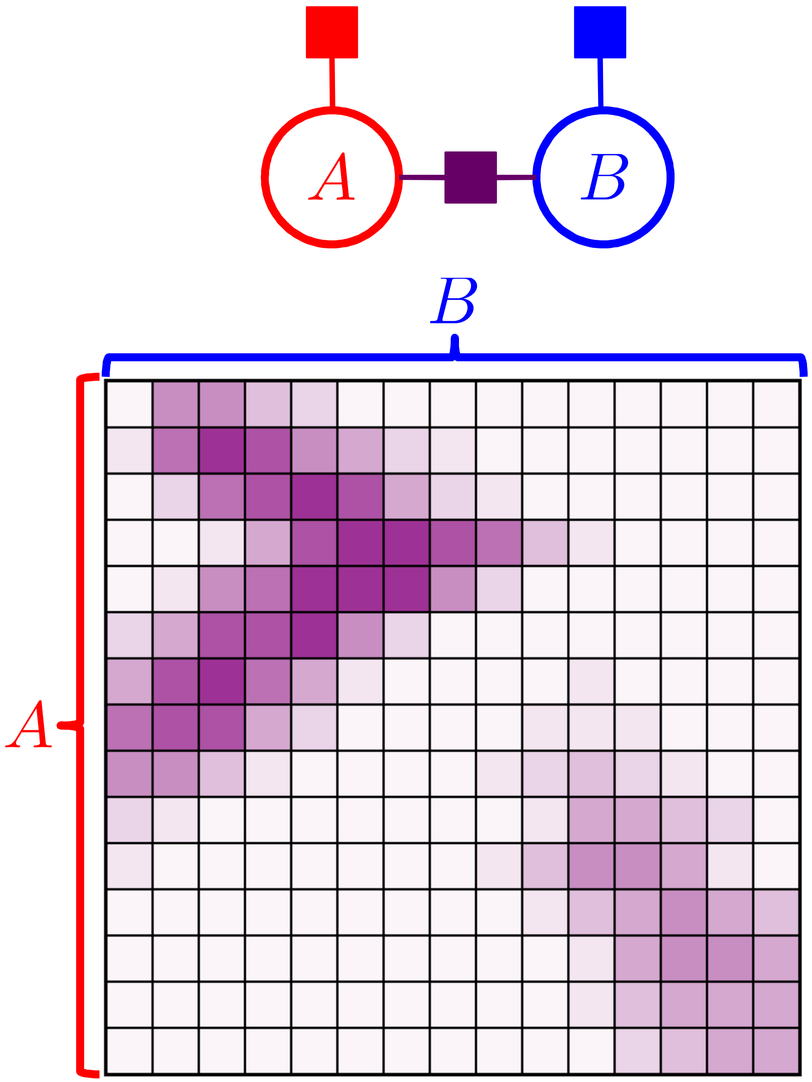
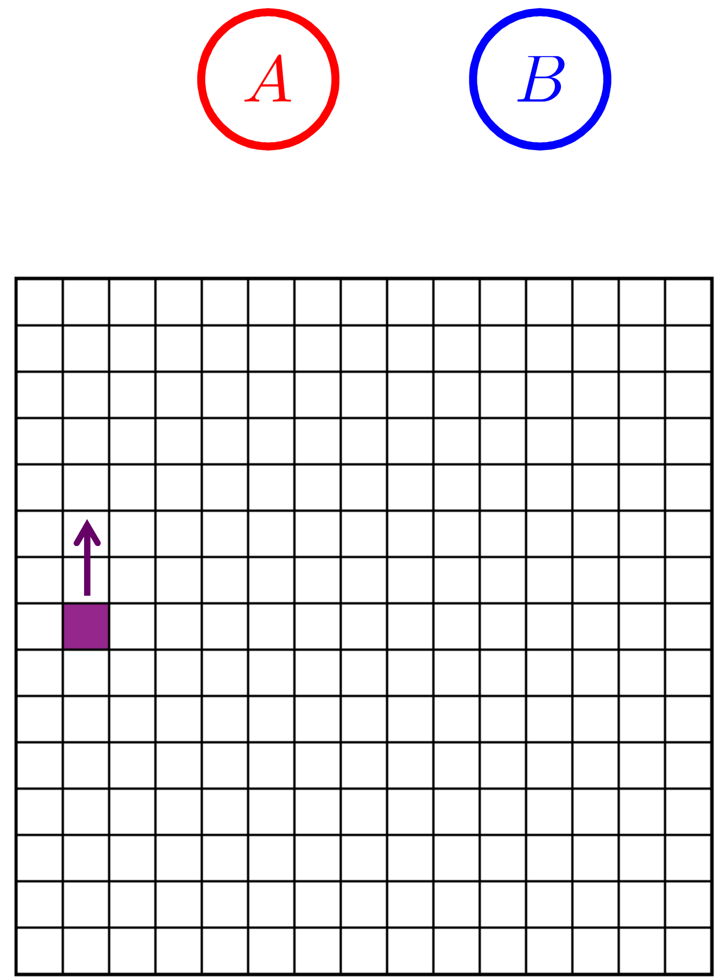
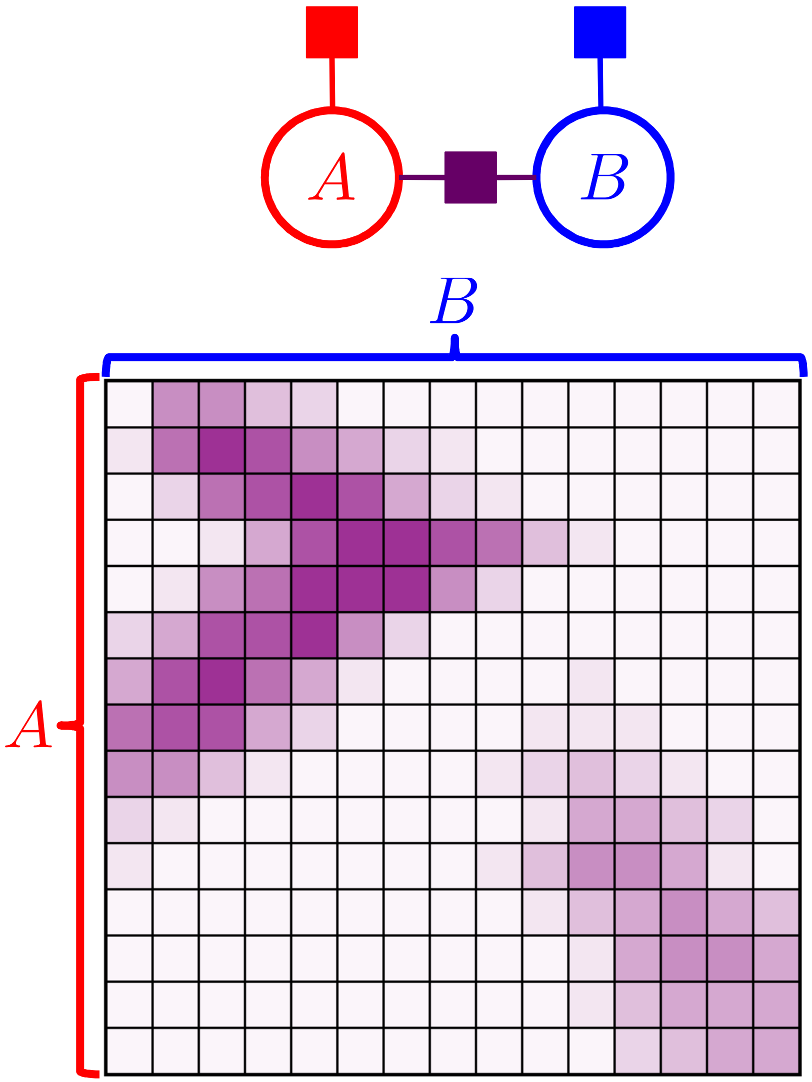
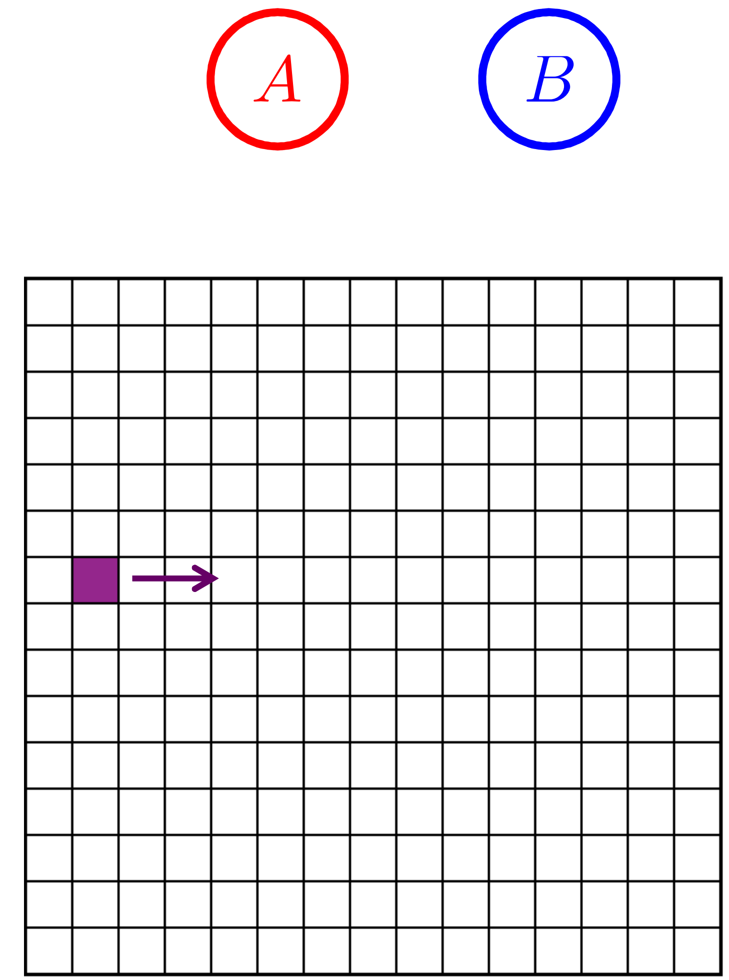


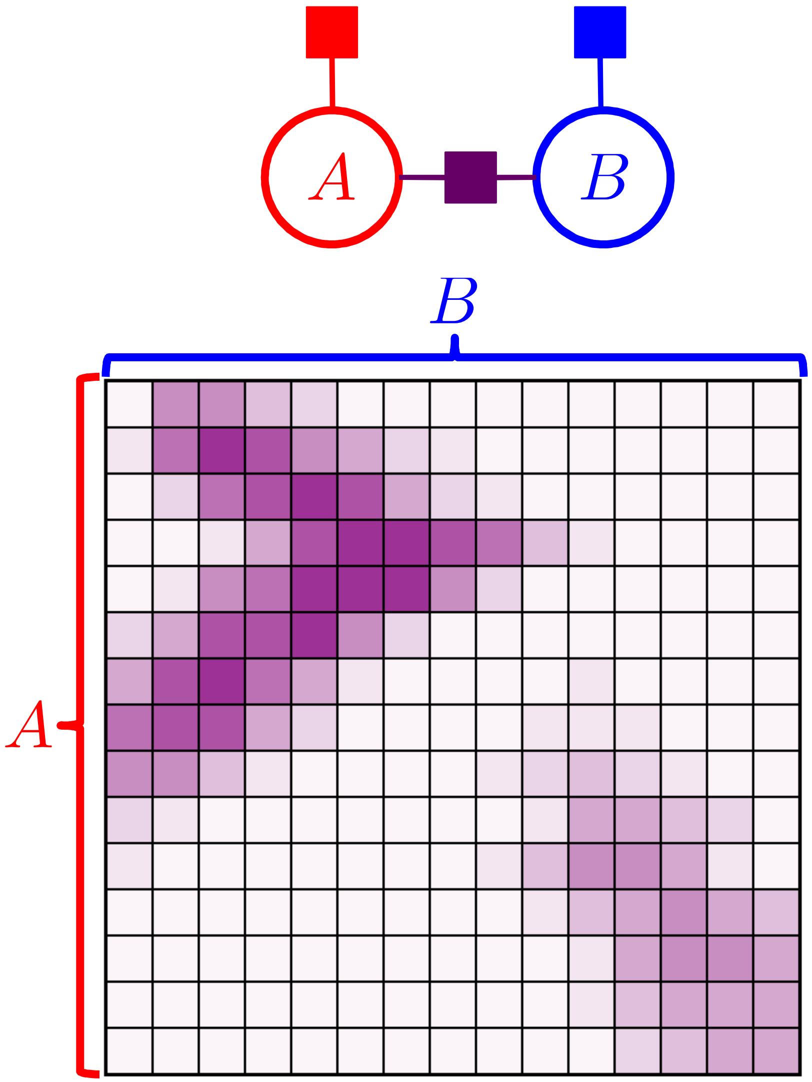


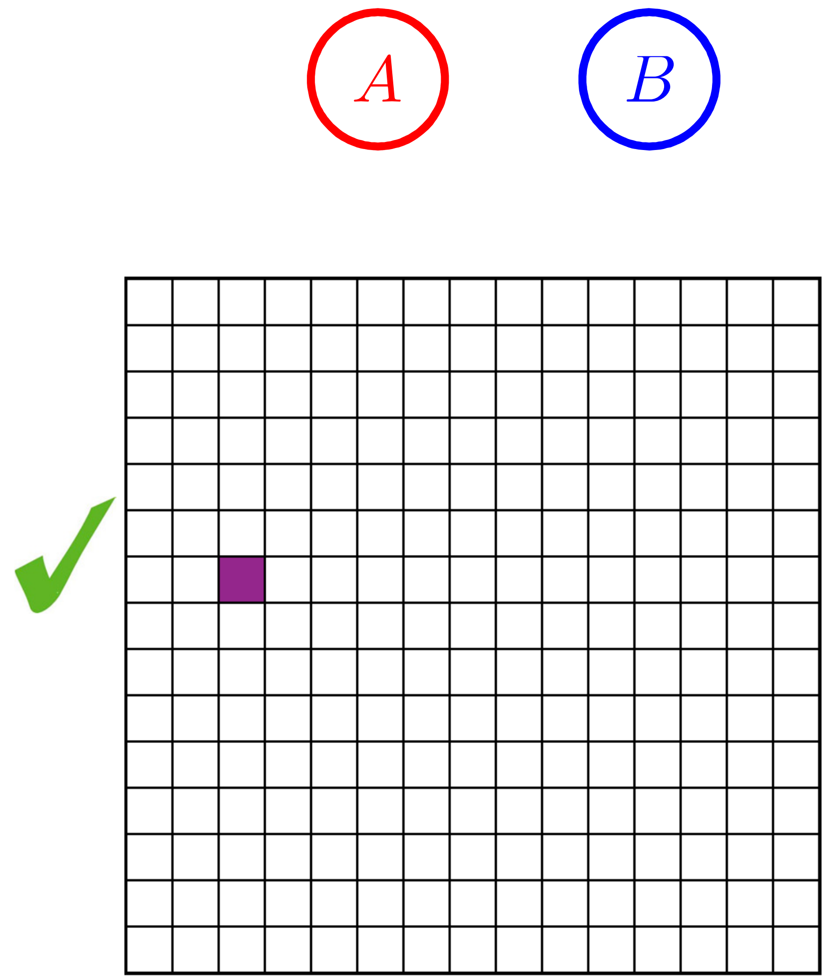
Iterated Conditional Modes Example



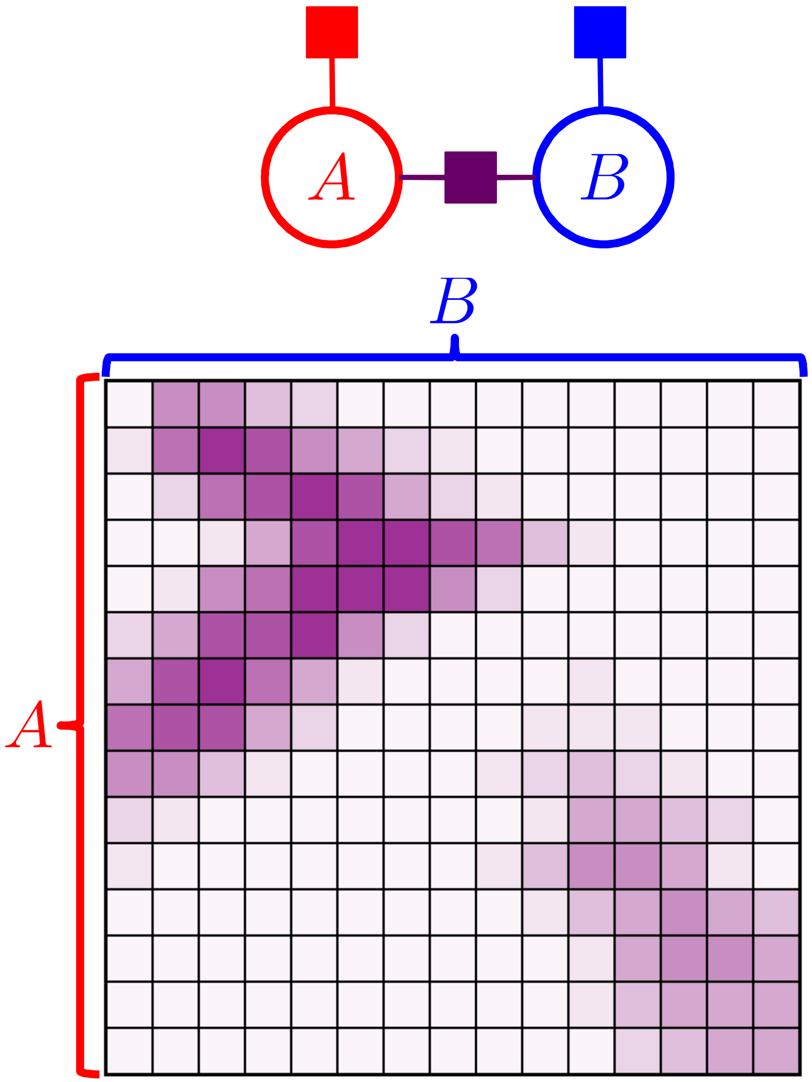


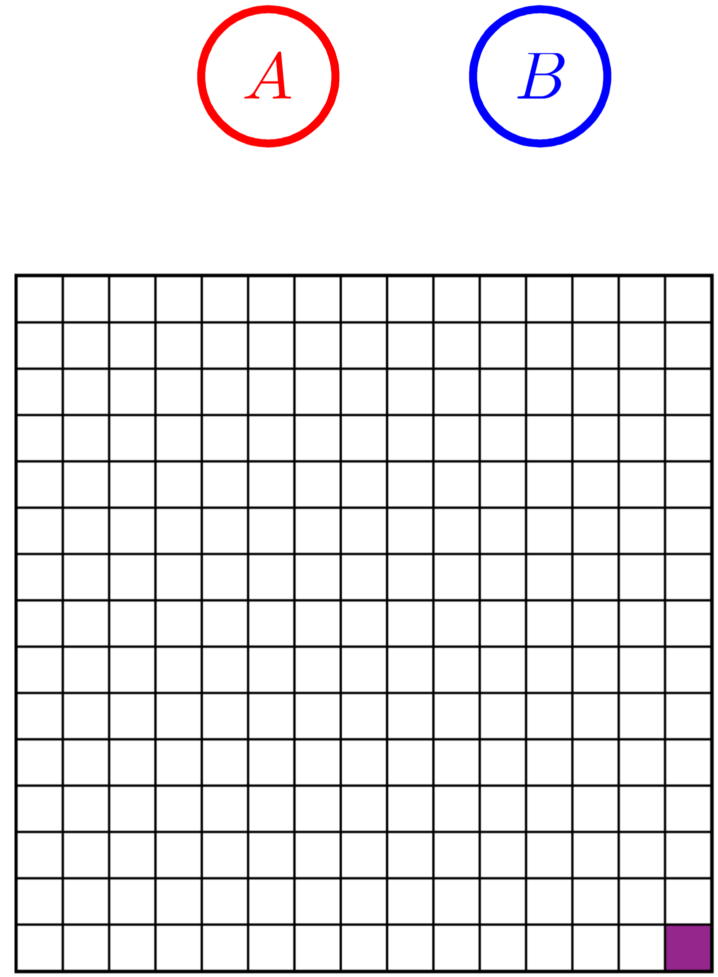
Iterated Conditional Modes Example



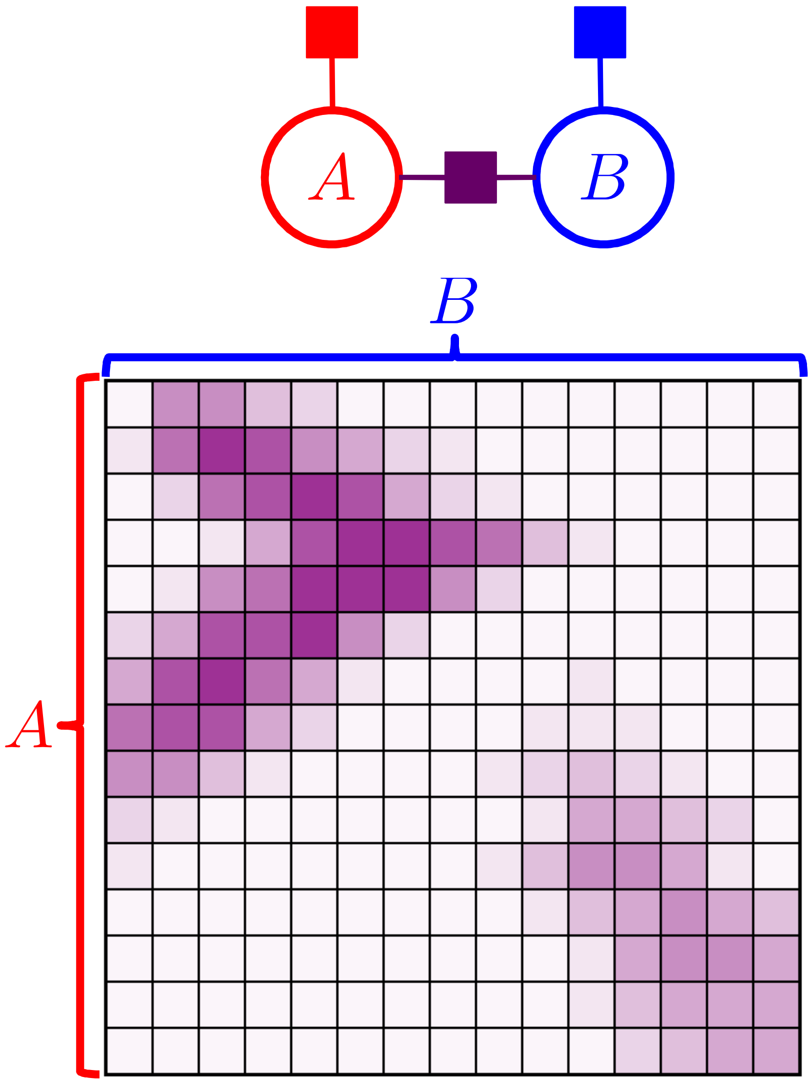


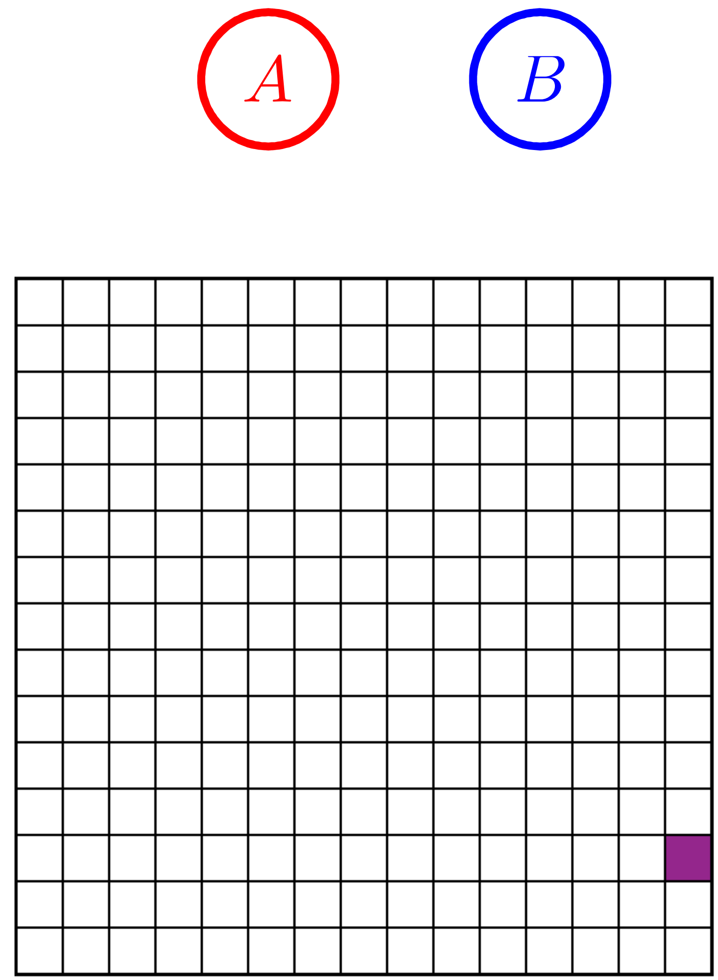
Iterated Conditional Modes Example



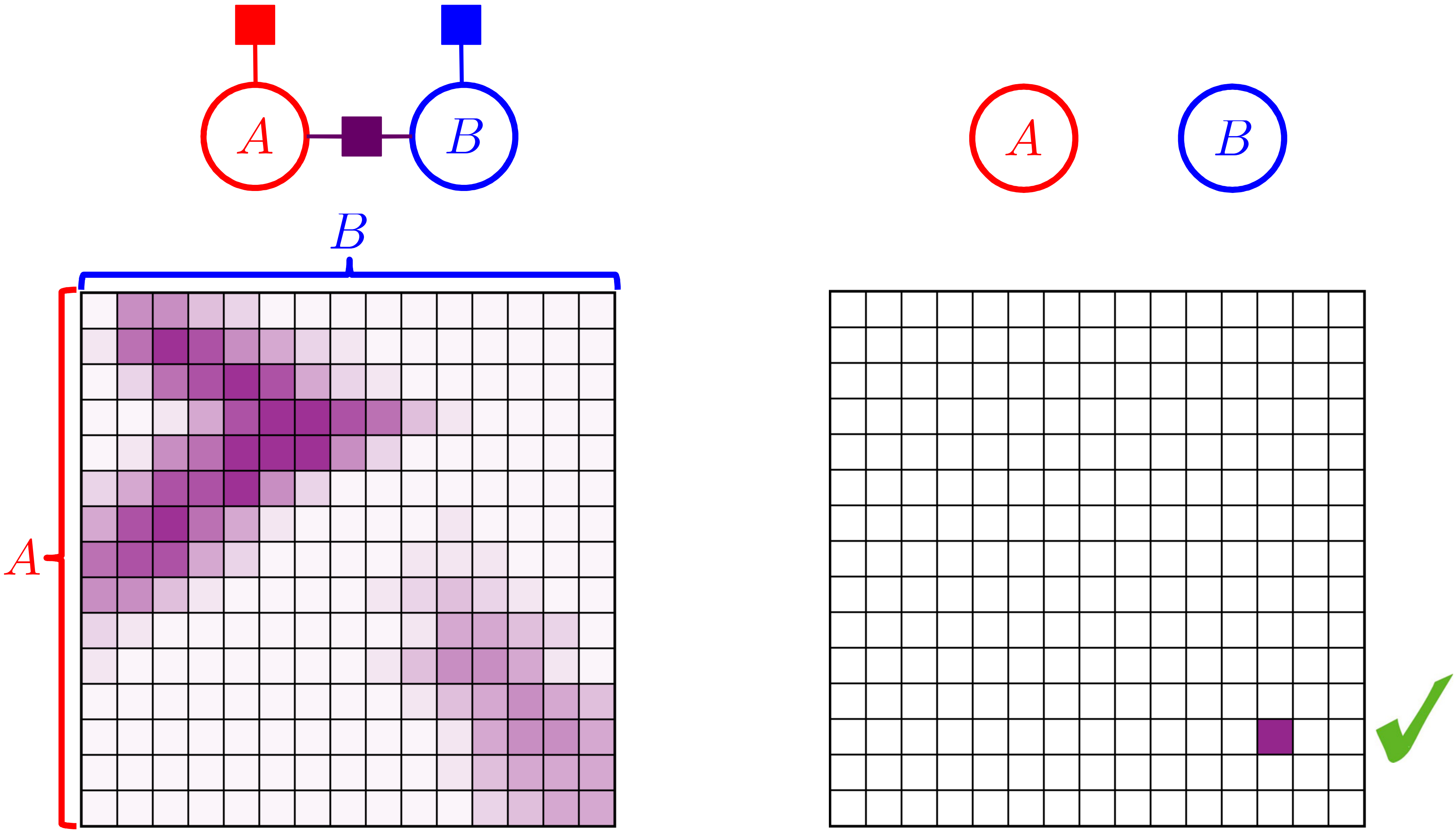


Iterated Conditional Modes Example





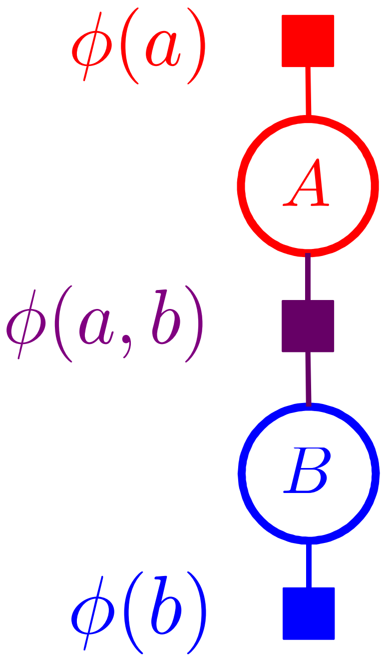
Iterated Conditional Modes Example



Mean Field Intro

Mean Field is coordinate ascent, just like Iterated Conditional Modes, but with soft assignments to each variable!

Mean Field Intro

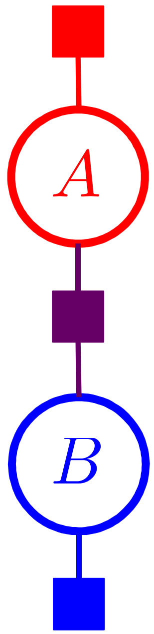


Wanted: $P(a|x)$, $P(b|x)$

Idea: coordinate ascent

Key object: (approx) marginals

Mean Field Intro



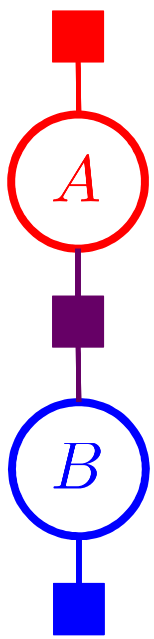
$$\phi(a) = \exp(w^\top f(a))$$

$$\phi(a,b) = \exp(w^\top f(a,b))$$

$$\phi(b) = \exp(w^\top f(b))$$

$$P(a, b|x) \propto \phi(a)\phi(b)\phi(a,b)$$

Mean Field Intro



$$\phi(a) = \exp(w^\top f(a))$$

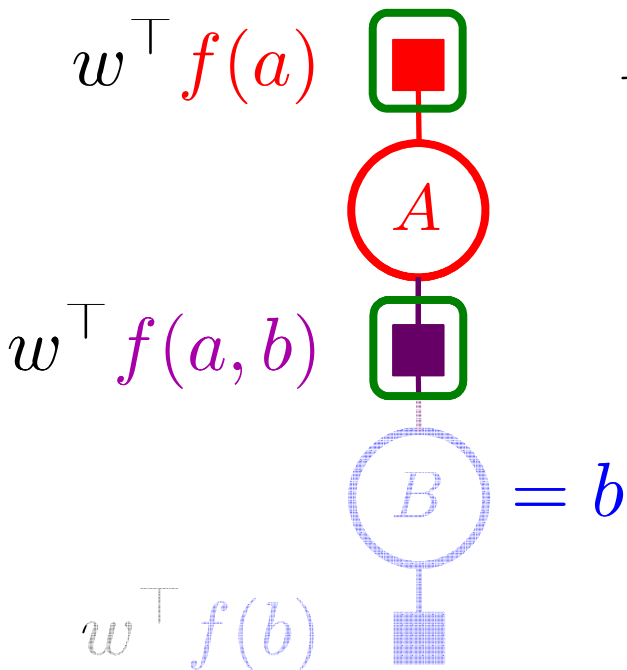
$$\phi(a,b) = \exp(w^\top f(a,b))$$

$$\phi(b) = \exp(w^\top f(b))$$

$$P(a,b|x) \propto \phi(a)\phi(b)\phi(a,b)$$

$$= \exp(w^\top f(a) + w^\top f(b) + w^\top f(a,b))$$

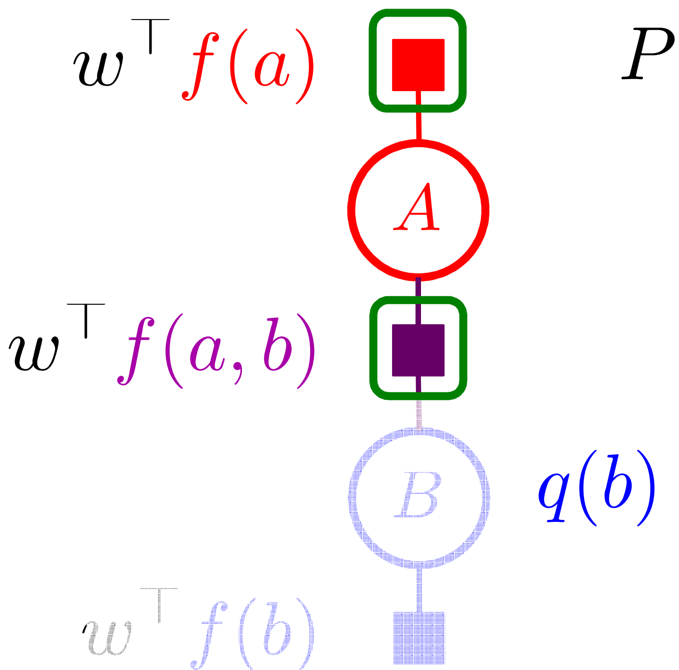
Mean Field Intro



$$P(a|b,x) \propto \exp(w^\top f(a) + w^\top f(a,b))$$

Wanted: $P(a|x)$, $P(b|x)$

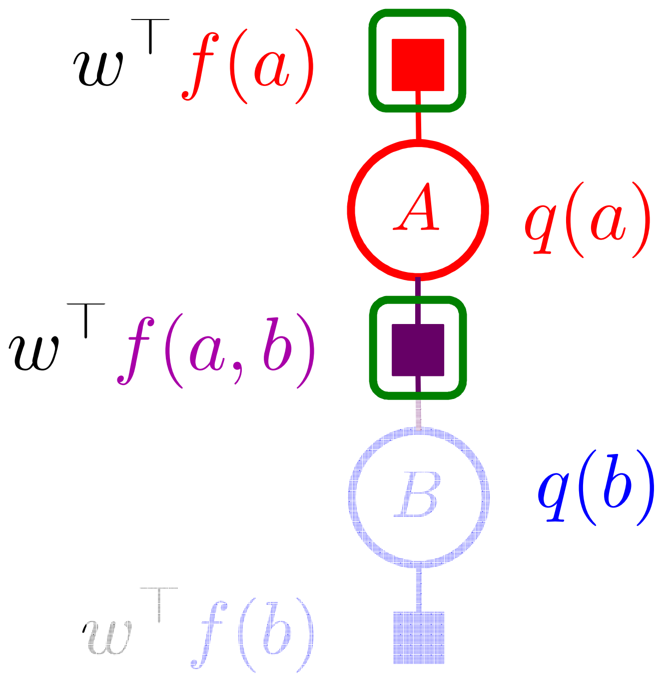
Mean Field Intro



$$P(a|b,x) \propto \exp(w^\top f(a) + w^\top f(a,b))$$

Wanted: $P(a|x)$, $P(b|x)$

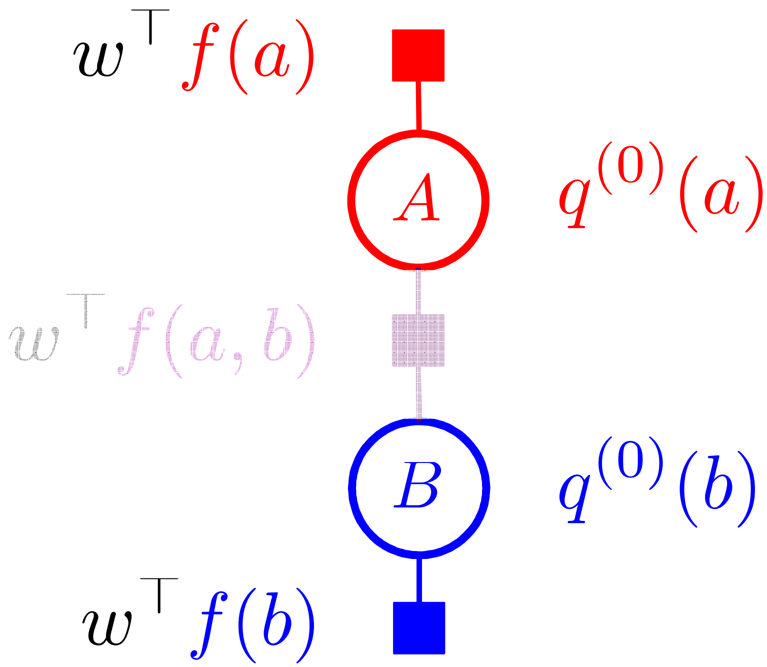
Mean Field Intro



$$q(a) \propto \exp\left(w^\top f(a) + w^\top \mathbb{E}_{q(b)} f(a,b)\right)$$

Wanted: $P(a|x)$, $P(b|x)$

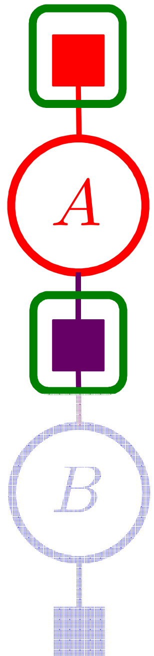
Mean Field Procedure



Wanted: $P(a|x), P(b|x)$

Mean Field Procedure

$w^\top f(a)$



$w^\top f(a,b)$

$w^\top f(b)$

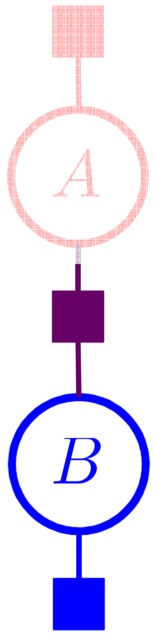
$$q^{(t)}(a) \propto \exp(w^\top f(a) + w^\top \mathbb{E}_{q(b)} f(a,b))$$

$q^{(t-1)}(b)$

Wanted: $P(a|x)$, $P(b|x)$

Mean Field Procedure

$w^\top f(a)$



$q^{(t)}(a)$

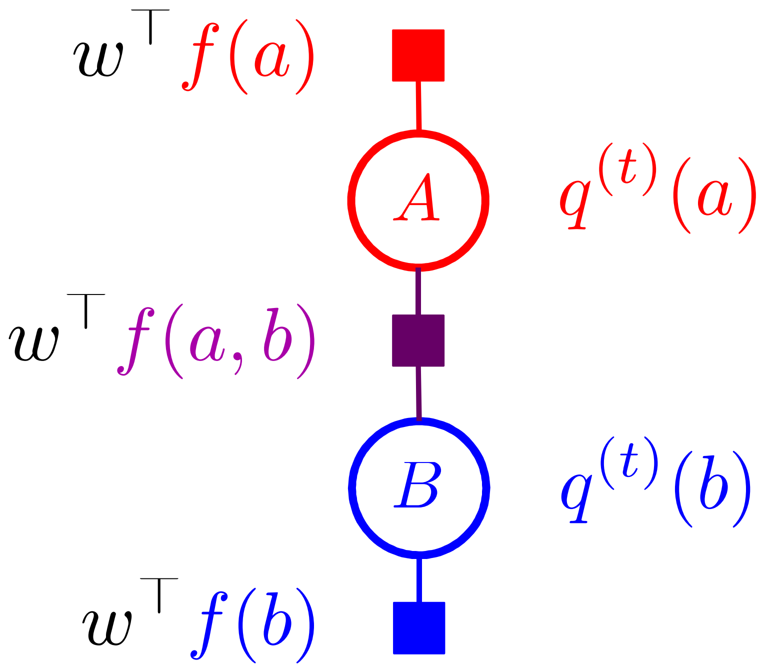
$w^\top f(a,b)$

$$q^{(t)}(b) \propto \exp(w^\top f(b) + w^\top \mathbb{E}_{q(a)} f(a,b))$$

$w^\top f(b)$

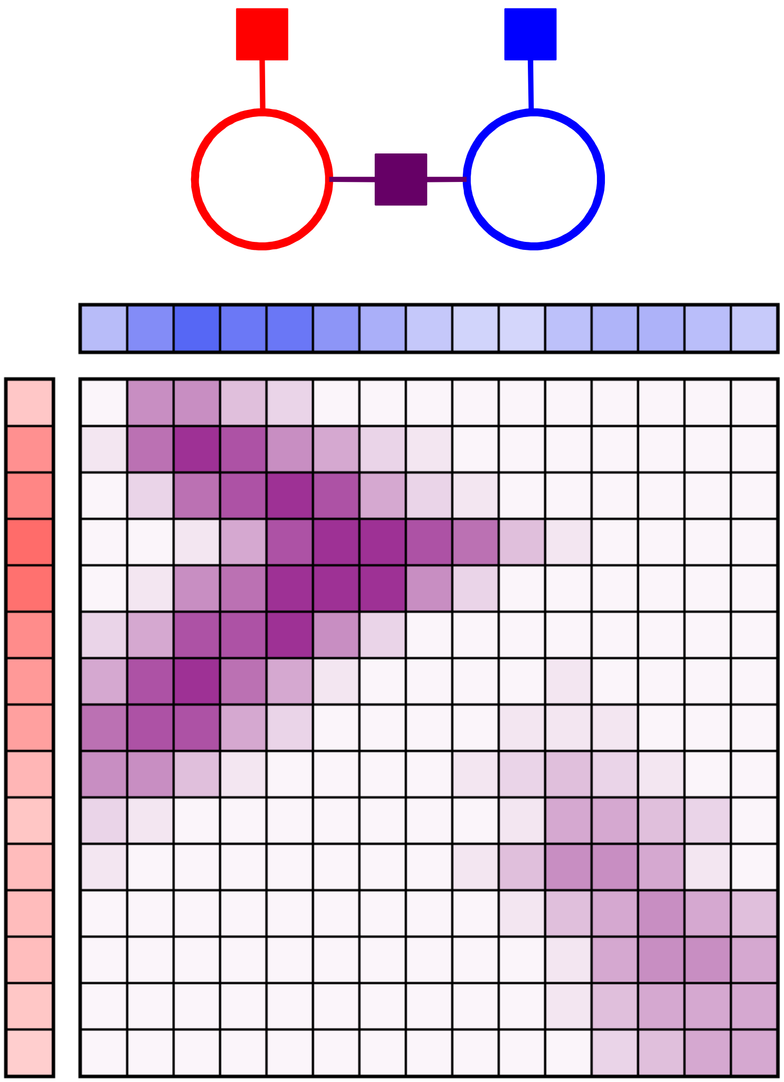
Wanted: $P(a|x), P(b|x)$

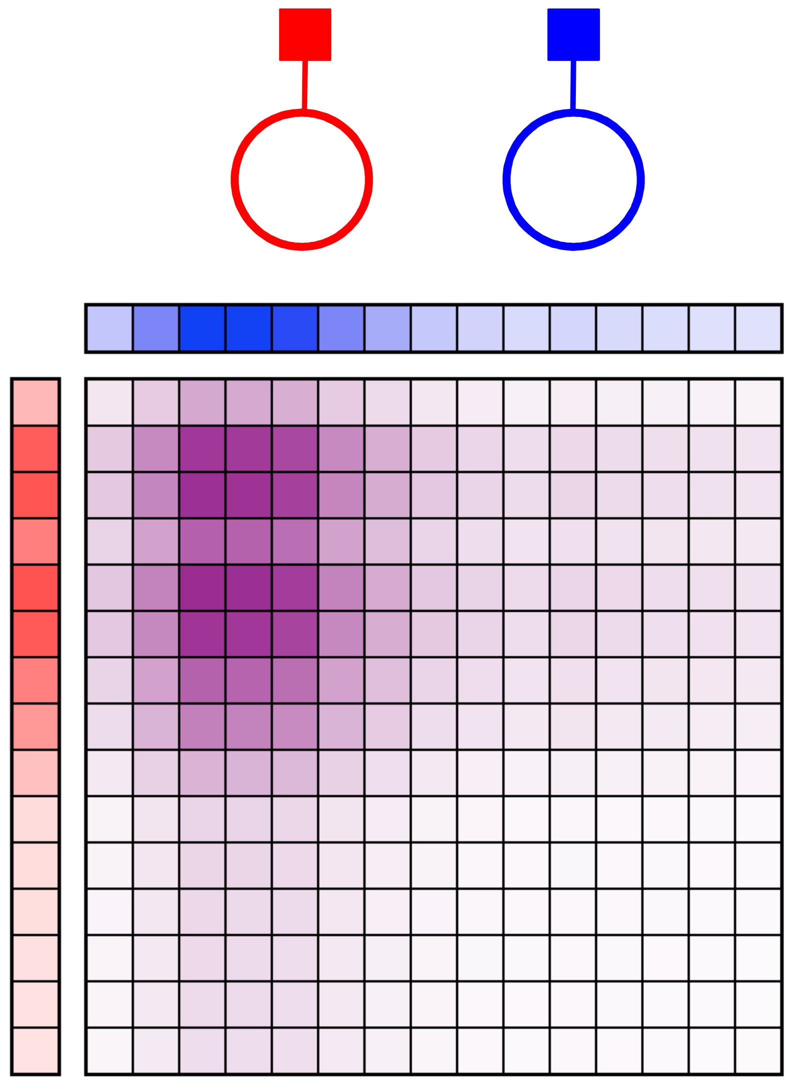
Mean Field Procedure



Wanted: $P(a|x), P(b|x)$

Example Results





Mean Field Derivation

- Goal: $$p(y) = P(y|x) \propto \exp\left(\sum_c w^\top f(y_c)\right)$$
- Approximation: $q(y) \approx p(y)$
- Constraint: $$q(y) = \prod_i q(y_i)$$
- Objective: $$q(y) = \operatorname*{argmin}_q KL(q||p)$$
- Procedure: Coordinate ascent on each $q(y_i)$
- What's the update?

Mean Field Update

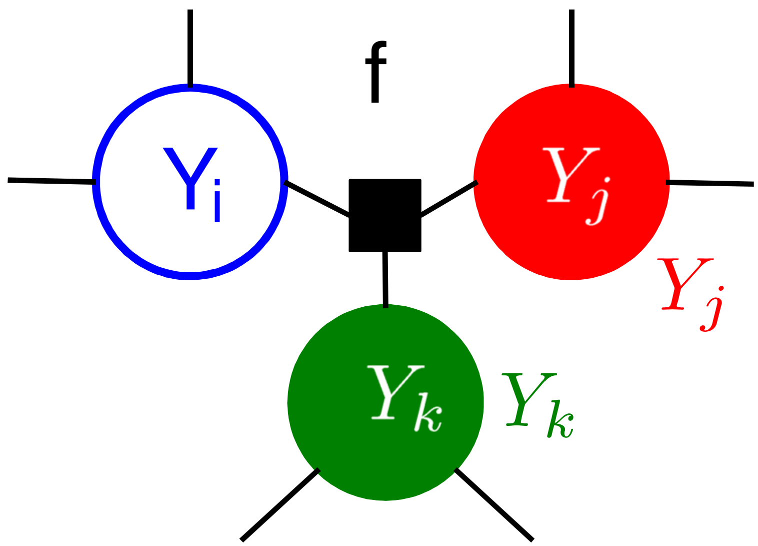
1) $$q(y_i) = \operatorname*{argmin}_{q(y_i)} KL(q||p)$$

2) $$\frac{\partial KL(q||p)}{\partial q(y_i)} = 0$$

3-9) Lots of algebra

10) $$q(y_i) \propto \exp\left(\sum_{c \in \mathcal{N}(i)} w^\top \mathbb{E}_{q(y_{-i})} f_c(y_c)\right)$$

Approximate Expectations



$$\mathbb{E}_{q(y_{-i})} f(y_i, y_j, y_k) = \sum_{y_j} \sum_{y_k} q(y_j) q(y_k) f(y_i, y_j, y_k)$$

General: $$\mathbb{E}_{q(y_{-i})} f_c(y_c) = \sum_{y_{c \setminus \{i\}}} \left(\prod_{j \in c \setminus \{i\}} q(y_j) \right) f_c(y_c)$$

General Update *

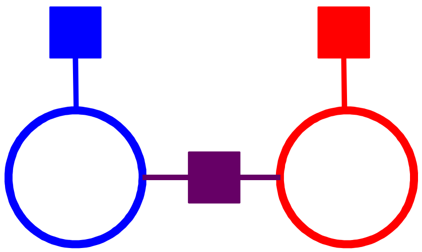
Exponential Family:

$$q(y_i) \propto \exp\left(\sum_{c \in \mathcal{N}(i)} w^\top \mathbb{E}_{q(y_{-i})} f_c(y_c)\right)$$

Generic:

$$q(y_i) \propto \exp\left(\sum_{c \in \mathcal{N}(i)} \mathbb{E}_{q(y_{-i})} \log \phi_c(y_c)\right)$$

Mean Field Inference Example



$\phi(y_2)$

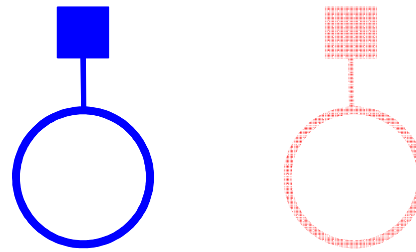
1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$



$q(y_2)$

.5	.5

$q(y_1)$

.5
.5

$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

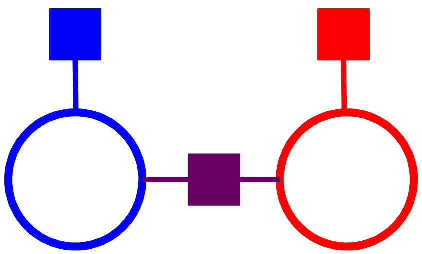
.2	.5
.2	.1

$p(y_1, y_2)$

$$q(Y_1 = 0) \propto \exp(0.50 \log 2 + 0.50 \log 5)$$

$$q(Y_1 = 1) \propto \exp(0.50 \log 2 + 0.50 \log 1)$$

Mean Field Inference Example



$\phi(y_2)$

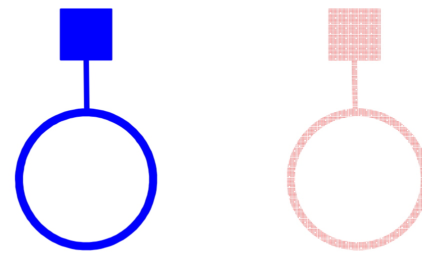
1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$



$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

.2	.5
.2	.1

$p(y_1, y_2)$

$q(y_2)$

.5	.5

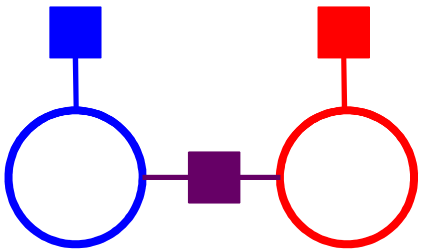
$q(y_1)$

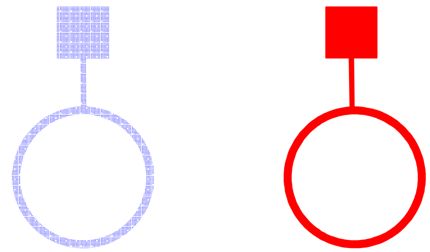
.69
.31

$$q(Y_1 = 0) \propto \exp(0.50 \log 2 + 0.50 \log 5)$$

$$q(Y_1 = 1) \propto \exp(0.50 \log 2 + 0.50 \log 1)$$

Mean Field Inference Example





$\phi(y_2)$

1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$

$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

.2	.5
.2	.1

$p(y_1, y_2)$

$q(y_2)$

.5	.5

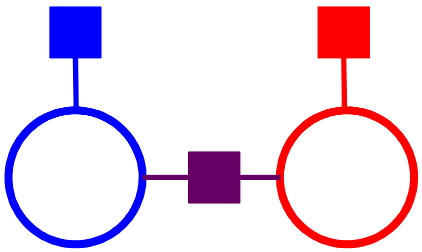
$q(y_1)$

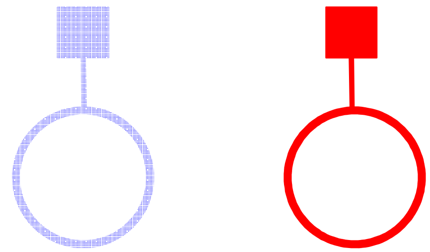
.69
.31

$$q(Y_2 = 0) \propto \exp(0.69 \log 2 + 0.31 \log 2)$$

$$q(Y_2 = 1) \propto \exp(0.69 \log 5 + 0.31 \log 1)$$

Mean Field Inference Example





$\phi(y_2)$

1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$

$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

.2	.5
.2	.1

$p(y_1, y_2)$

$q(y_2)$

.40	.60

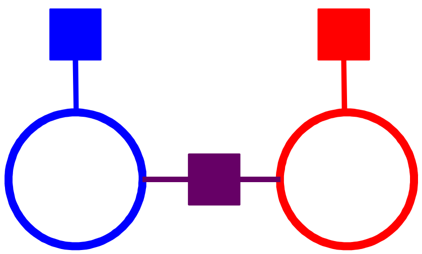
$q(y_1)$

.69
.31

$$q(Y_2 = 0) \propto \exp(0.69 \log 2 + 0.31 \log 2)$$

$$q(Y_2 = 1) \propto \exp(0.69 \log 5 + 0.31 \log 1)$$

Mean Field Inference Example



$\phi(y_2)$

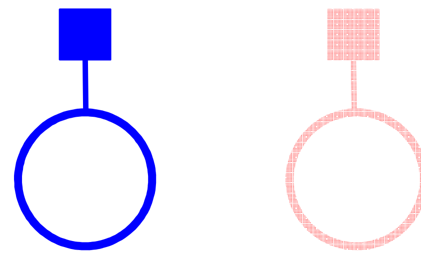
1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$



$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

.2	.5
.2	.1

$p(y_1, y_2)$

$$q(Y_1 = 0) \propto \exp(0.40 \log 2 + 0.60 \log 5)$$

$$q(Y_1 = 1) \propto \exp(0.40 \log 2 + 0.60 \log 1)$$

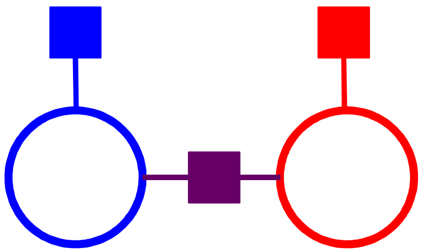
$q(y_2)$

.40	.60

$q(y_1)$

.73
.27

Mean Field Inference Example



$\phi(y_2)$

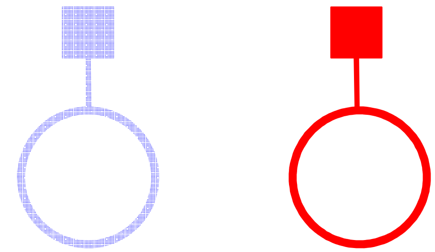
1	1

$\phi(y_1)$

1
1

$\phi(y_1, y_2)$

2	5
2	1



$p(y_2)$

.4	.6

$p(y_1)$

.7
.3

$p(y_1, y_2)$

.2	.5
.2	.1

$q(y_2)$

.38	.62

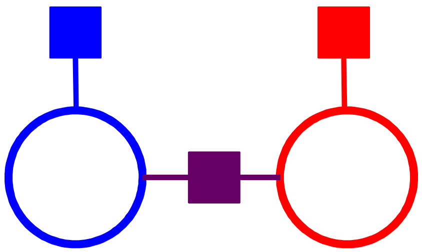
$q(y_1)$

.73
.27

$$q(Y_2 = 0) \propto \exp(0.73 \log 2 + 0.27 \log 2)$$

$$q(Y_2 = 1) \propto \exp(0.73 \log 5 + 0.27 \log 1)$$

Mean Field Inference Example



$\phi(y_2)$

1	1

$\phi(y_1)$

1
1

2	5
2	1

$\phi(y_1, y_2)$

$p(y_2)$

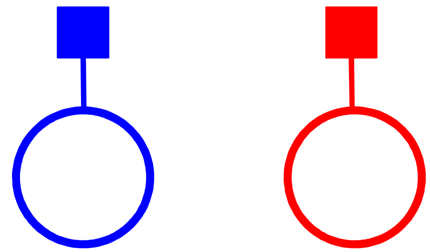
.4	.6

$p(y_1)$

.7
.3

.2	.5
.2	.1

$p(y_1, y_2)$



$q(y_2)$

.38	.62

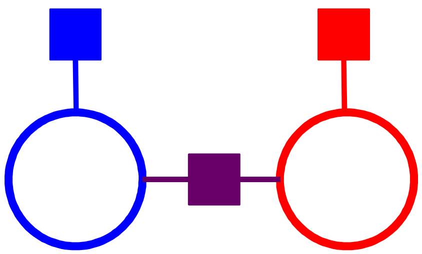
$q(y_1)$

.73
.27

.28	.45
.10	.17

$q(y_1, y_2)$

Mean Field Inference Example



$\phi(y_2)$

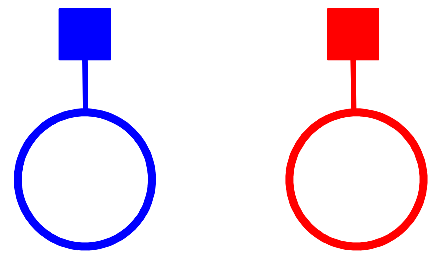
2	1

$\phi(y_1)$

2
1

1	1
1	1

$\phi(y_1, y_2)$



$p(y_2)$

.67	.33

$p(y_1)$

.67
.33

.44	.22
.22	.11

$p(y_1, y_2)$

$q(y_2)$

.67	.33

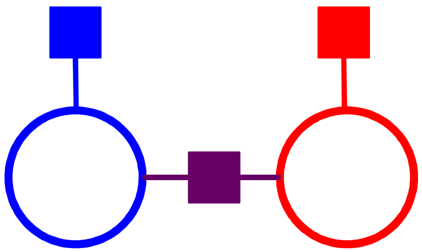
$q(y_1)$

.67
.33

.44	.22
.22	.11

$q(y_1, y_2)$

Mean Field Inference Example



$\phi(y_2)$

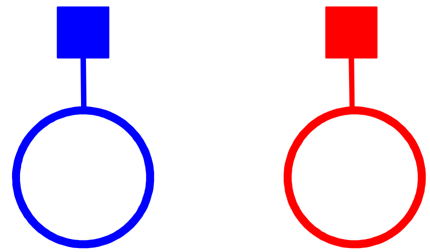
1	1

$\phi(y_1)$

1
1

9	1
1	5

$\phi(y_1, y_2)$



$p(y_2)$

.62	.38

$p(y_1)$

.62
.38

.56	.06
.06	.31

$p(y_1, y_2)$

$q(y_2)$

.82	.18

$q(y_1)$

.82
.18

.67	.15
.15	.03

$q(y_1, y_2)$

Mean Field Q&A

- Are the marginals guaranteed to converge to the right thing, like in sampling?

  No

- Is the algorithm at least guaranteed to converge to something?

  Yes

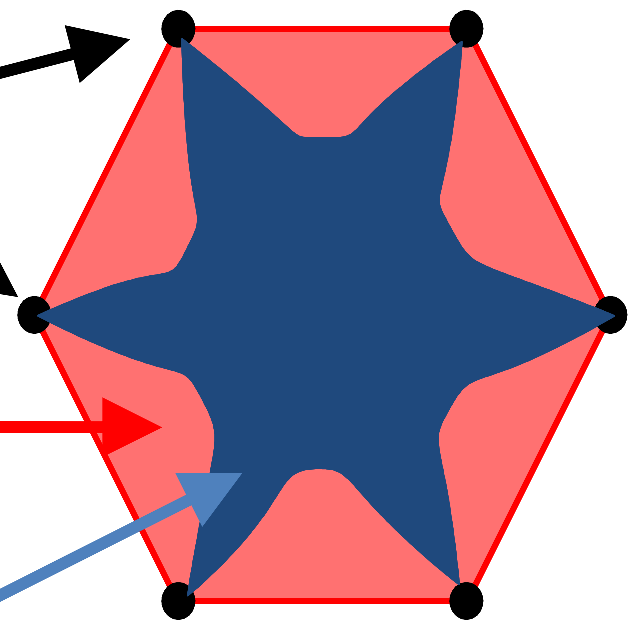
- So it's just like EM?

  Yes

Why Only Local Optima?!

Variables: $Y_1, Y_2, \ldots Y_n$

Discrete distributions:
e.g. P(0,1,0,…0) = 1



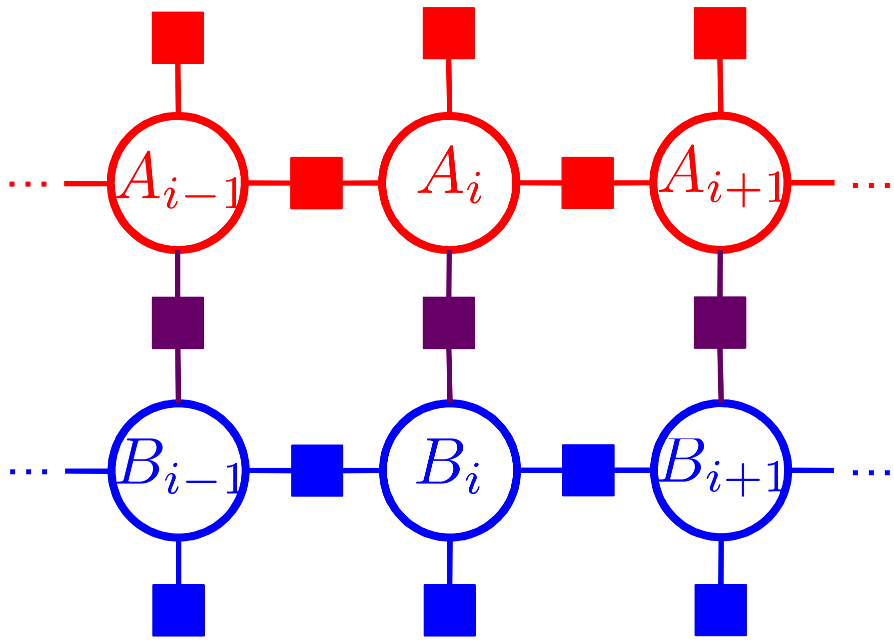
All distributions
(all convex combos)

Mean field approximable
(can represent all discrete ones, but not all)

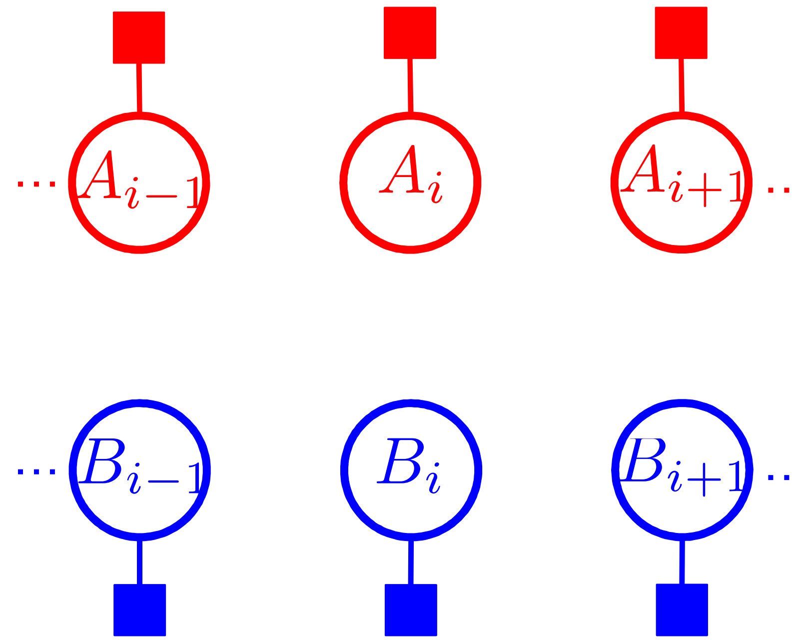
Part 3: Structured Mean Field

Mean Field Approximation

Model:



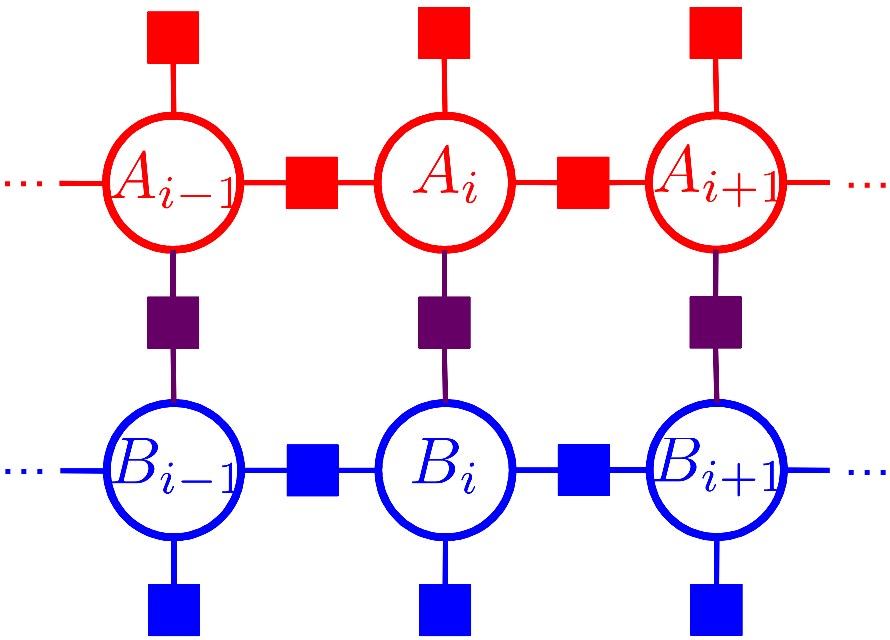
Approximate Graph:



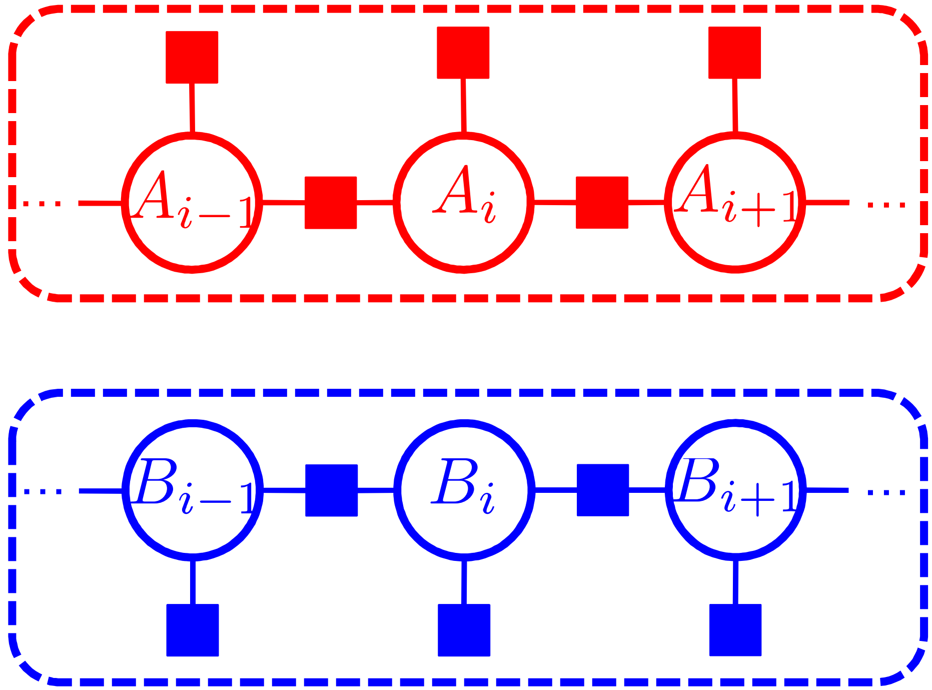
$$p(a,b) \approx \prod_i q(a_i)q(b_i)$$

Structured Mean Field Approximation

Model:



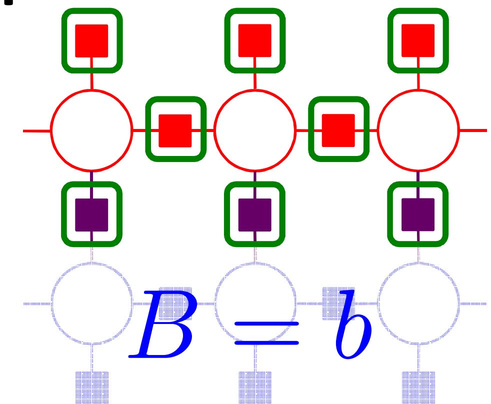
Approximate Graph:



$$p(a, b) \approx q(a)q(b)$$

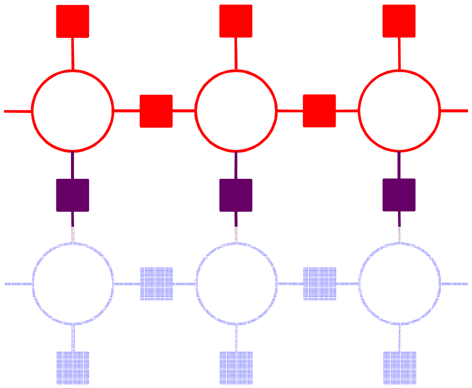
(Xing et al, 2003)

Structured Mean Field Approximation



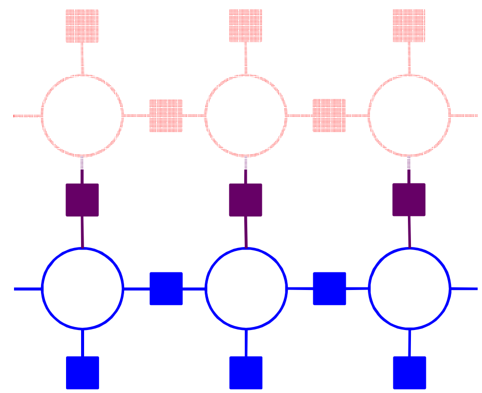
$$P(a|b,x) \propto \exp\left(\sum_i w^\top f(a_i) + \sum_i w^\top f(a_{i-1}, a_i) + \sum_i w^\top f(a_i, b_i)\right)$$

Structured Mean Field Approximation



$$q(a) \propto \exp\left(\sum_i w^\top f(a_i) + \sum_i w^\top f(a_{i-1}, a_i) + \sum_i w^\top \mathbb{E}_{q(b)} f(a_i, b_i)\right)$$

Structured Mean Field Approximation



$$q(b) \propto \exp\left(\sum_i w^\top f(b_i) + \sum_i w^\top f(b_{i-1}, b_i) + \sum_i w^\top \mathbb{E}_{q(a)} f(a_i, b_i) \right)$$

Computing Structured Updates

$$q(a) \propto \exp\left(\sum_i w^\top f(a_i) + \sum_i w^\top f(a_{i-1}, a_i) + \sum_i w^\top \mathbb{E}_{q(b)} f(a_i, b_i)\right)$$

w ✓

$f(a_i)$ ✓

$f(a_{i-1}, a_i)$ ✓

$\mathbb{E}_{q(b)} f(a_i, b_i)$??

Computing Structured Updates

$$\mathbb{E}_{q(b)} f(a_i, b_i) = \sum_{b} q(b) f(a_i, b_i)$$

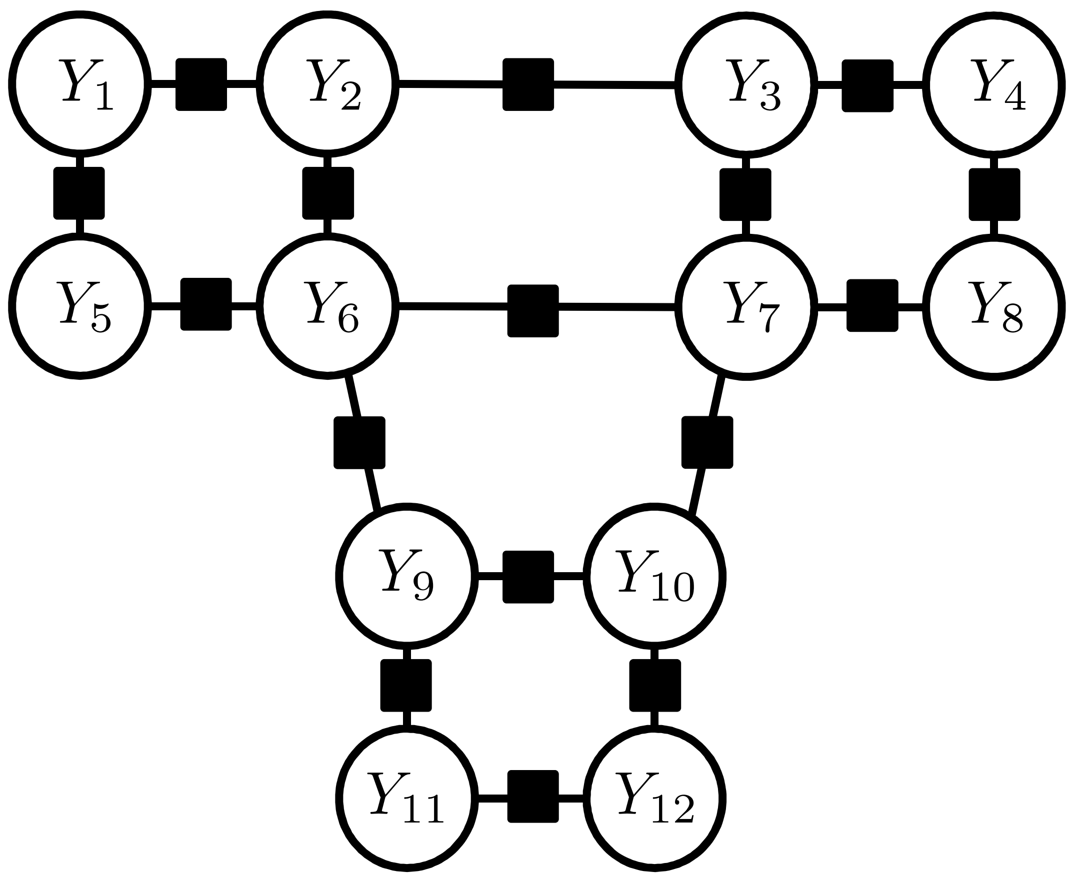
$$= \sum_{b_i} q(b_i) f(a_i, b_i)$$

Marginal probability of b_i under $q(b)$

Computed with forward-backward

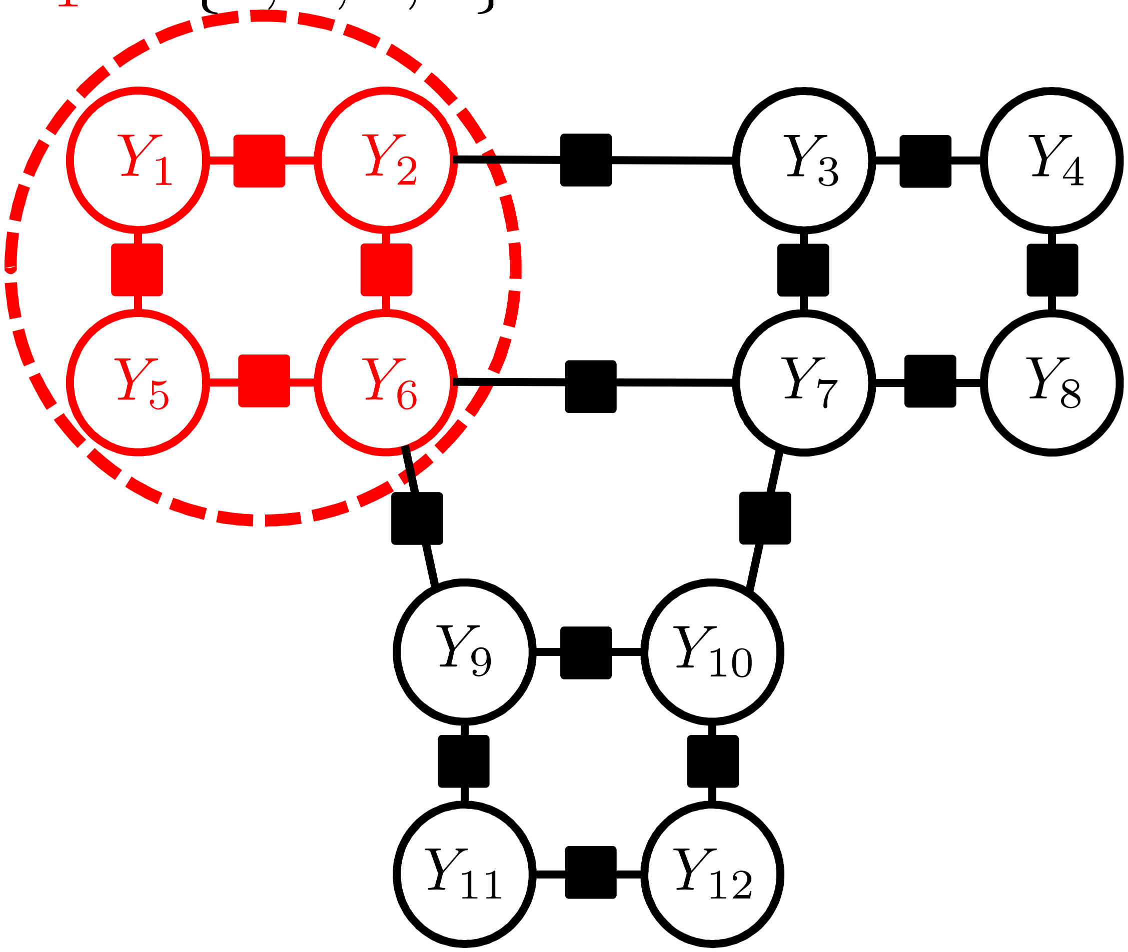
Updating $q(a)$ consists of computing all marginals $q(b_i)$

Structured Mean Field Notation



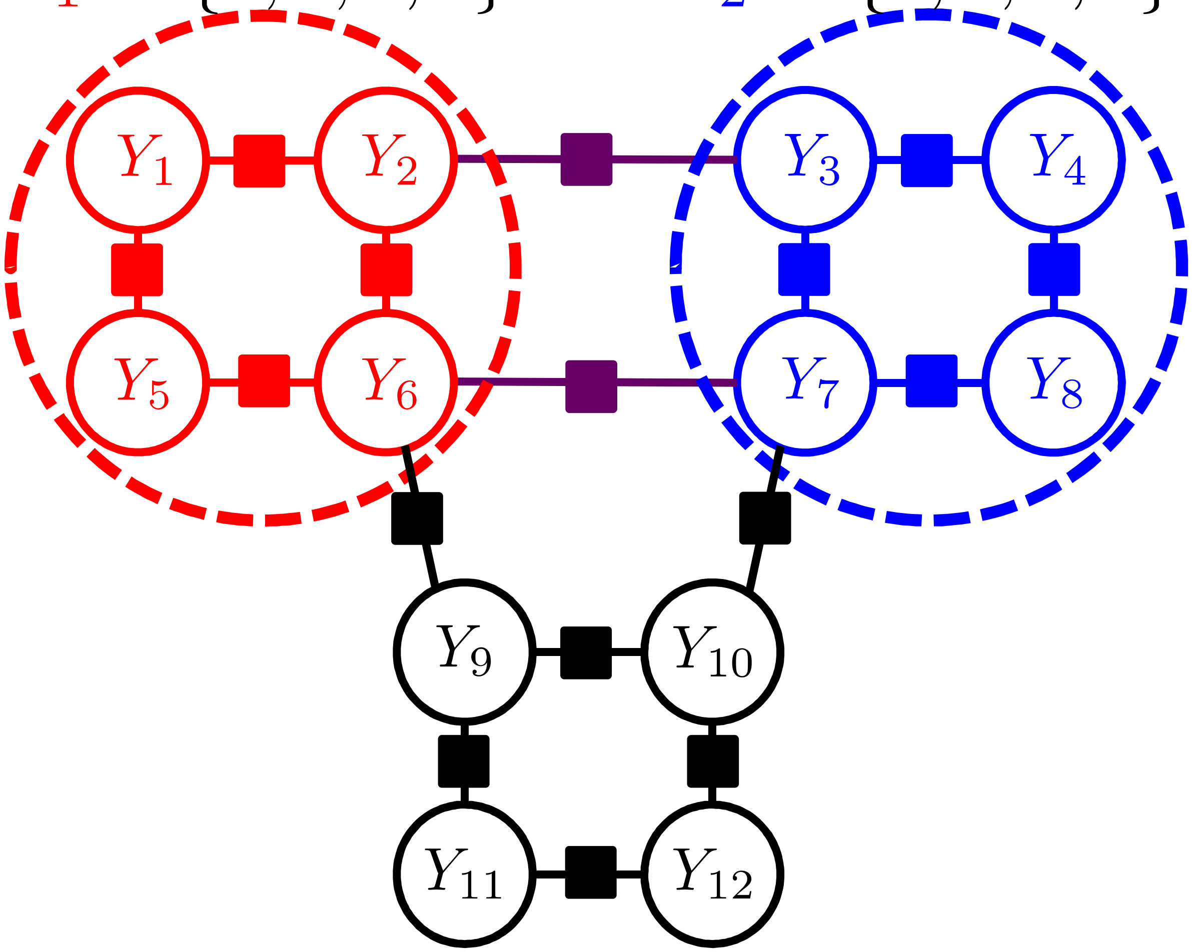
Structured Mean Field Notation

$d_1 = \{1, 2, 5, 6\}$

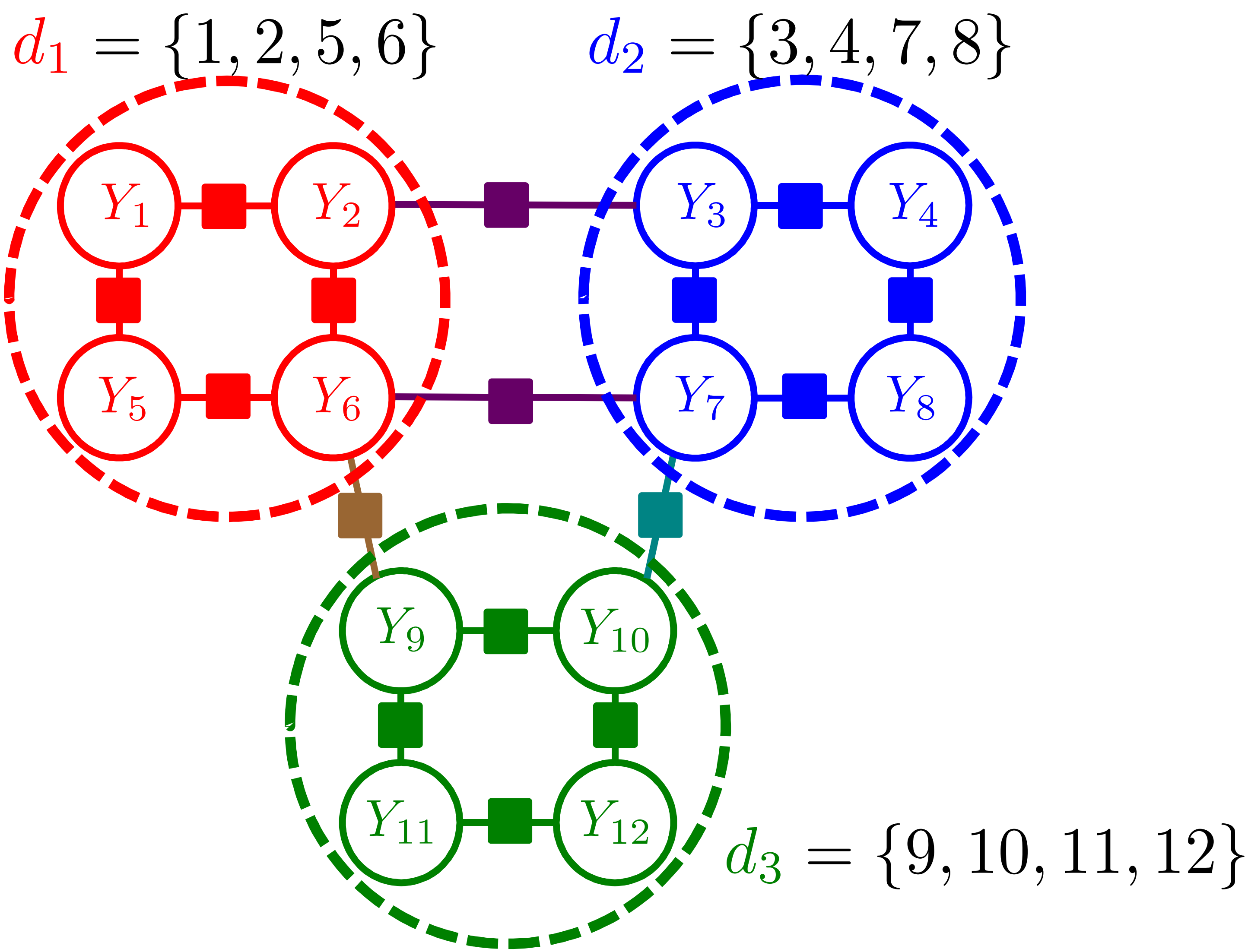


Structured Mean Field Notation

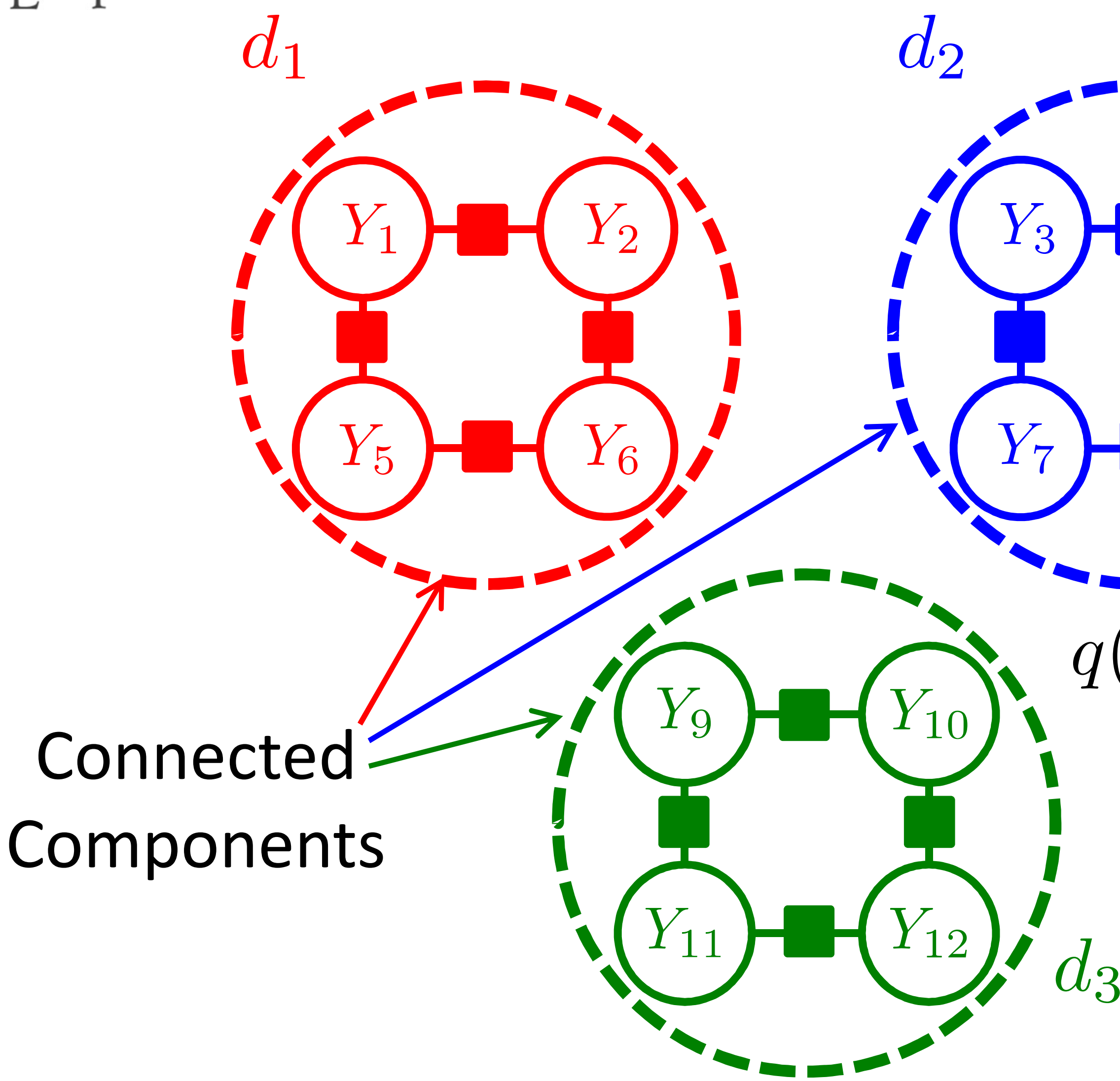
$d_1 = \{1, 2, 5, 6\}$ $d_2 = \{3, 4, 7, 8\}$



Structured Mean Field Notation

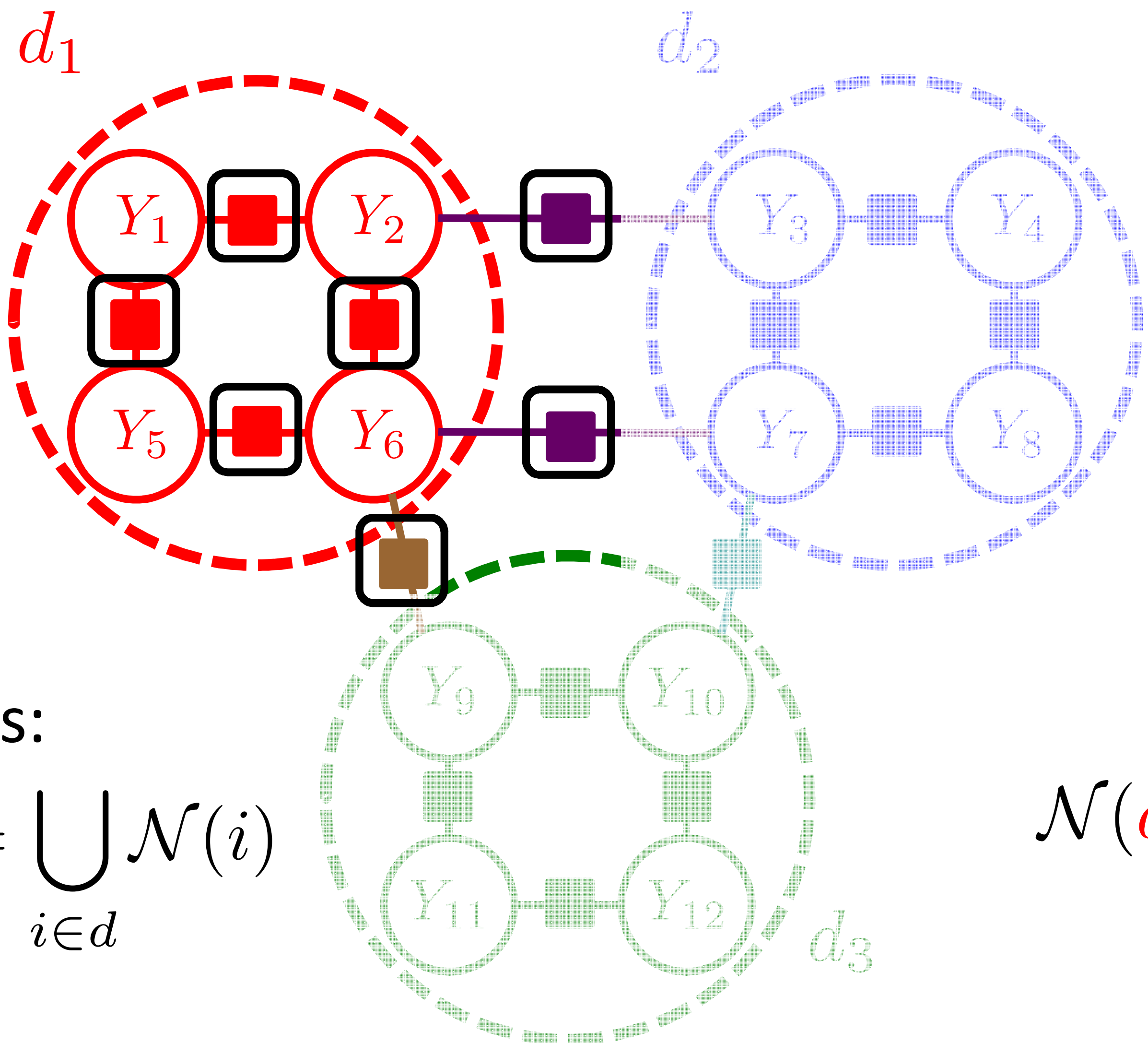


Structured Mean Field Notation



$$
\begin{aligned}
q(y) &= \prod_d q(y_d) \\
&= q(y_1, y_2, y_5, y_6) \cdot \\
&\quad\; q(y_3, y_4, y_7, y_8) \cdot \\
&\quad\; q(y_9, y_{10}, y_{11}, y_{12})
\end{aligned}
$$

Structured Mean Field Notation



Neighbors:

$$\mathcal{N}(d) = \bigcup_{i \in d} \mathcal{N}(i)$$

$$\mathcal{N}(d_1)$$

Structured Mean Field Updates

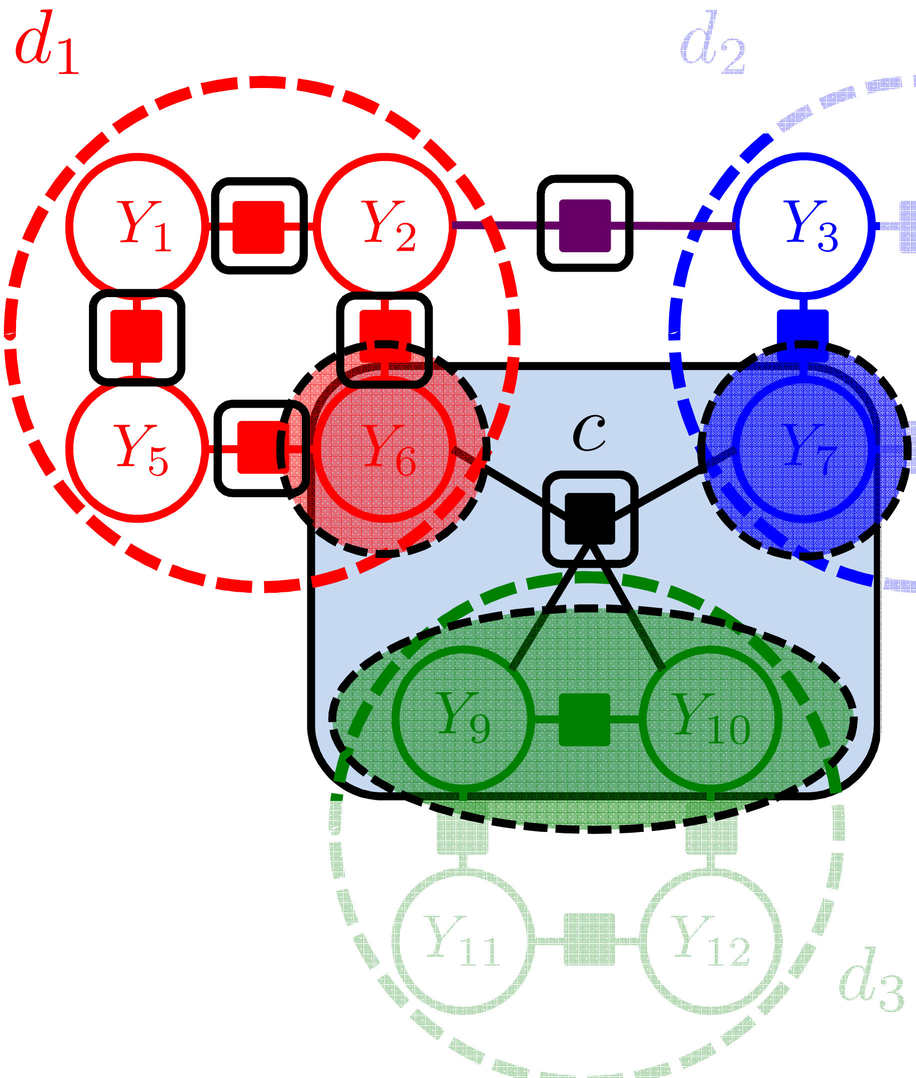
Naïve Mean Field:

$$q(y_i) \propto \exp\left(\sum_{c \in \mathcal{N}(i)} w^\top \mathbb{E}_{q_{-i}} f(y_c)\right)$$

Structured Mean Field:

$$q(y_d) \propto \exp\left(\sum_{c \in \mathcal{N}(d)} w^\top \mathbb{E}_{q_{-d}} f(y_c)\right)$$

Expected Feature Counts



$q(d_1)$

$\mathcal{N}(d_1)$

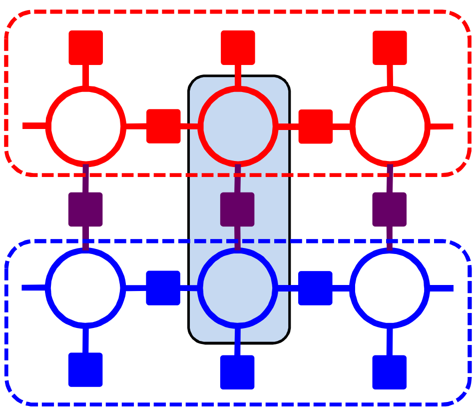
$$\mathbb{E}_{q_{-d_1}} f(y_6, y_7, y_9, y_{10})$$

$$= \sum_{y_7, y_9, y_{10}} f(y_6, y_7, y_9, y_{10})$$

Component Factorizability *

Condition

$$f(a_i, b_i) = f(a_i)f(b_i)$$



Example Feature

$$f(a_i, b_i) = \begin{cases} 1 & a_i = \text{NNP} \ \& \\ & b_i = \text{B-PER} \\ 0 & \text{otherwise} \end{cases}$$

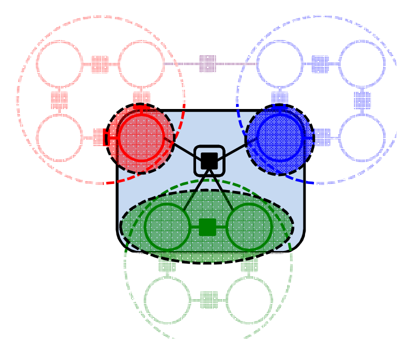
$$= \left(\begin{cases} 1 & a_i = \text{NNP} \\ 0 & \text{otherwise} \end{cases} \right) \cdot \left(\begin{cases} 1 & b_i = \text{B-PER} \\ 0 & \text{otherwise} \end{cases} \right)$$

$$= f(a_i)f(b_i)$$

Generic Condition

$$f_c(y_c) = \prod_{d : c \cap d \neq \emptyset} f_{c \cap d}(y_{c \cap d})$$

(pointwise product)

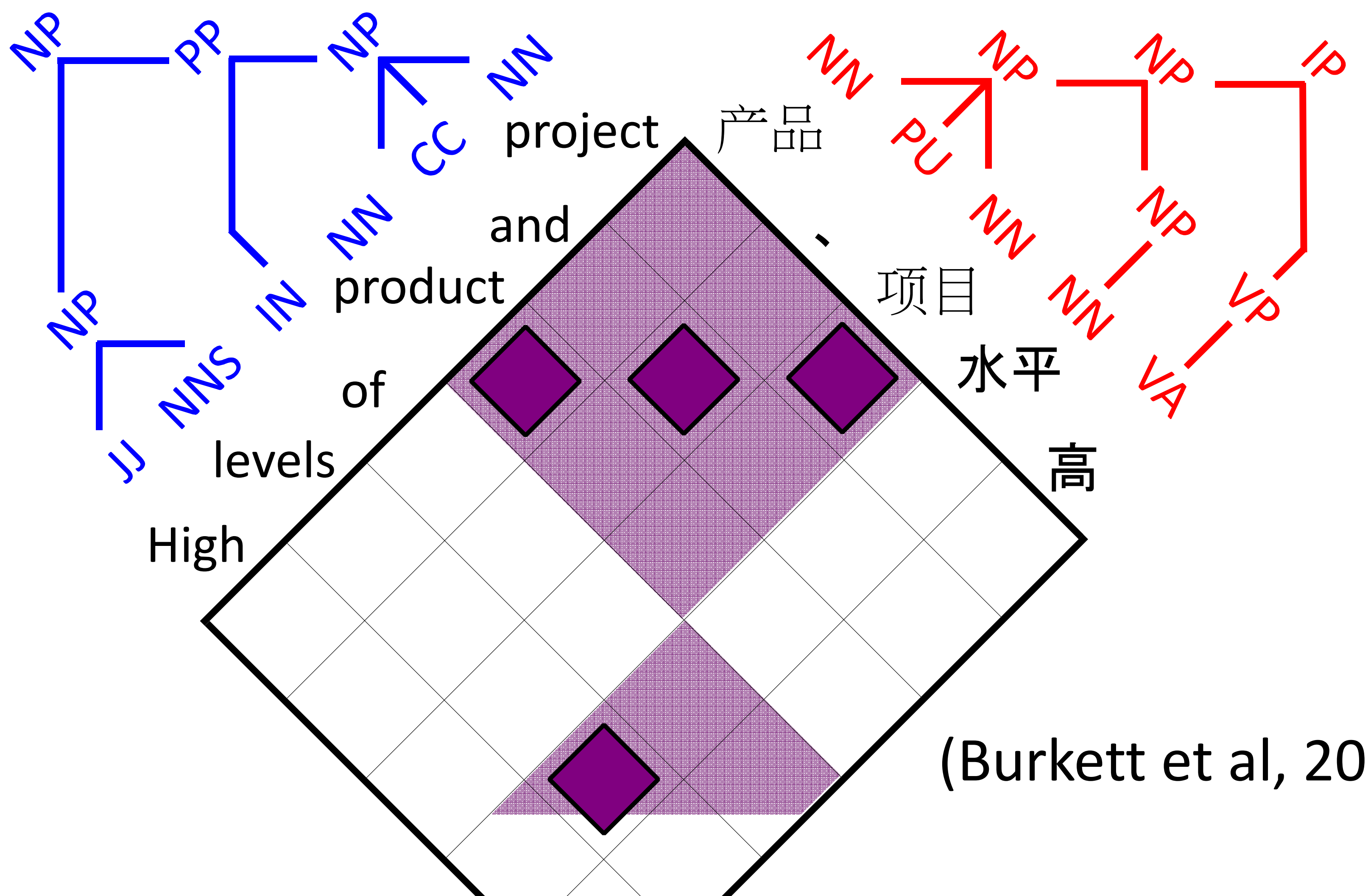


Component Factorizability *

(Abridged)

Use conjunctive indicator features

Joint Parsing and Alignment



(Burkett et al, 2010)

Joint Parsing and Alignment

Input:

Sentences

$$(s, s')$$

project 产品

and 、

product 项目

of 水平

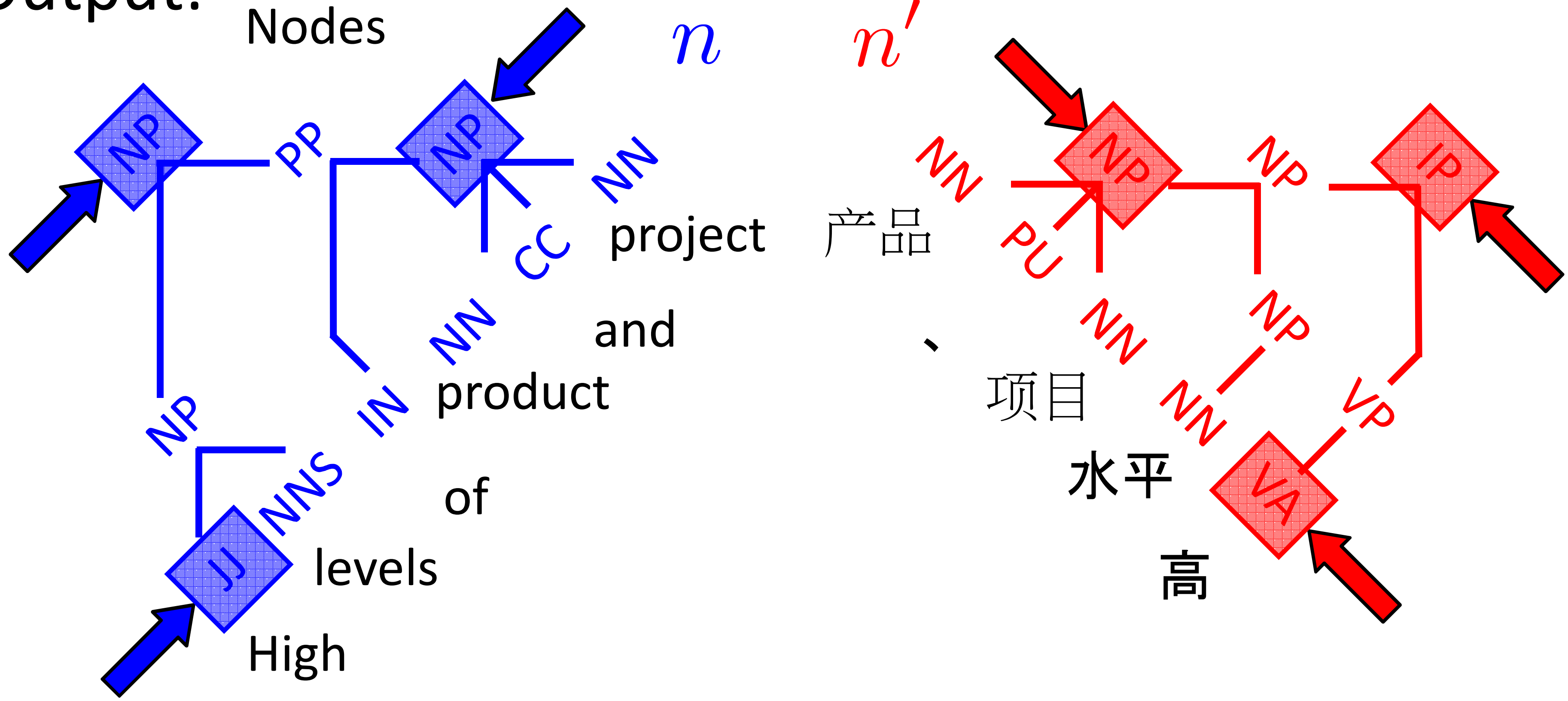
levels 高

High

Joint Parsing and Alignment

Output: Trees contain t t'

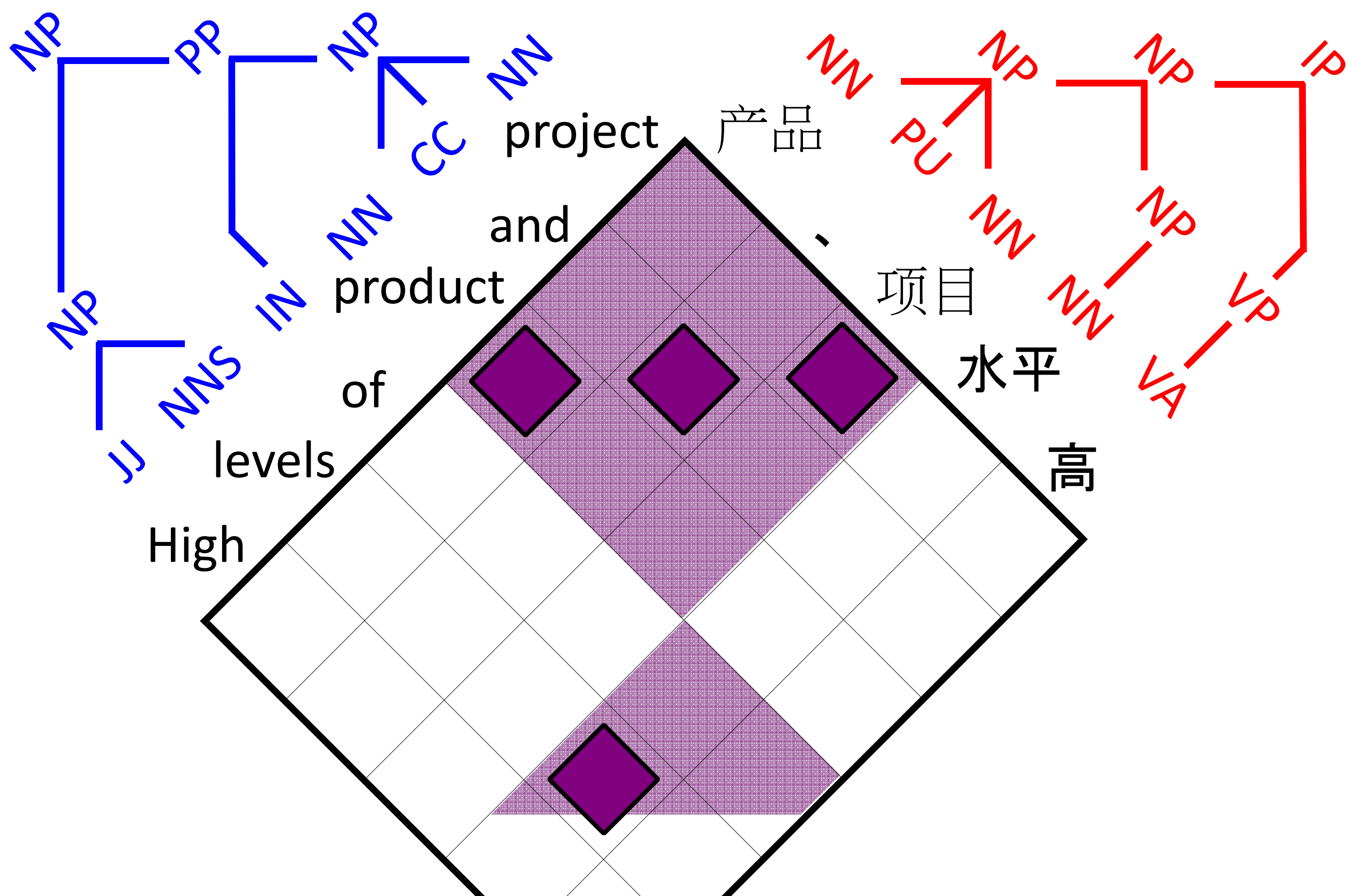
Nodes n n'



Joint Parsing and Alignment

Output:

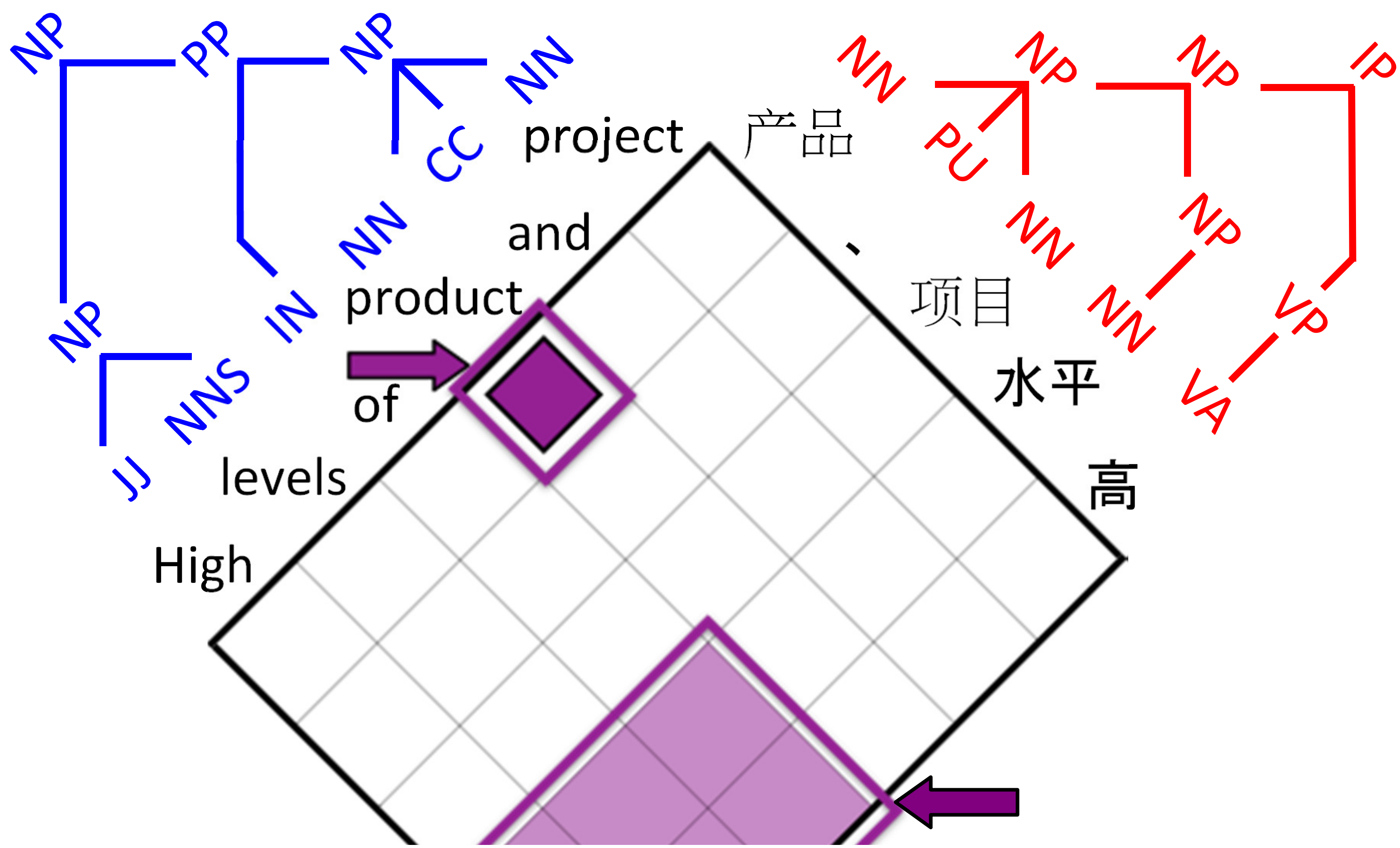
Alignments a



Joint Parsing and Alignment

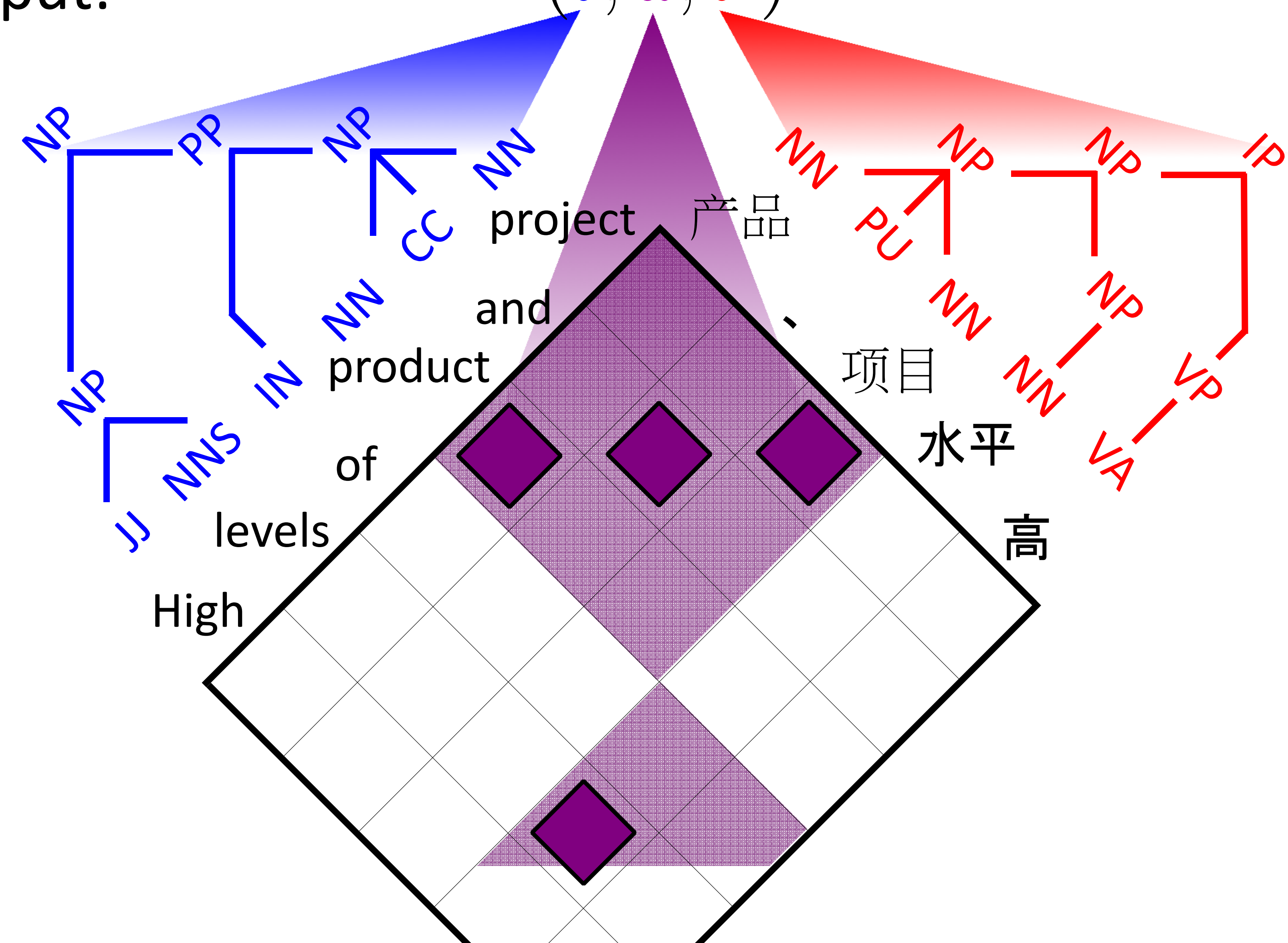
Output: Alignments a

contain Bispans b



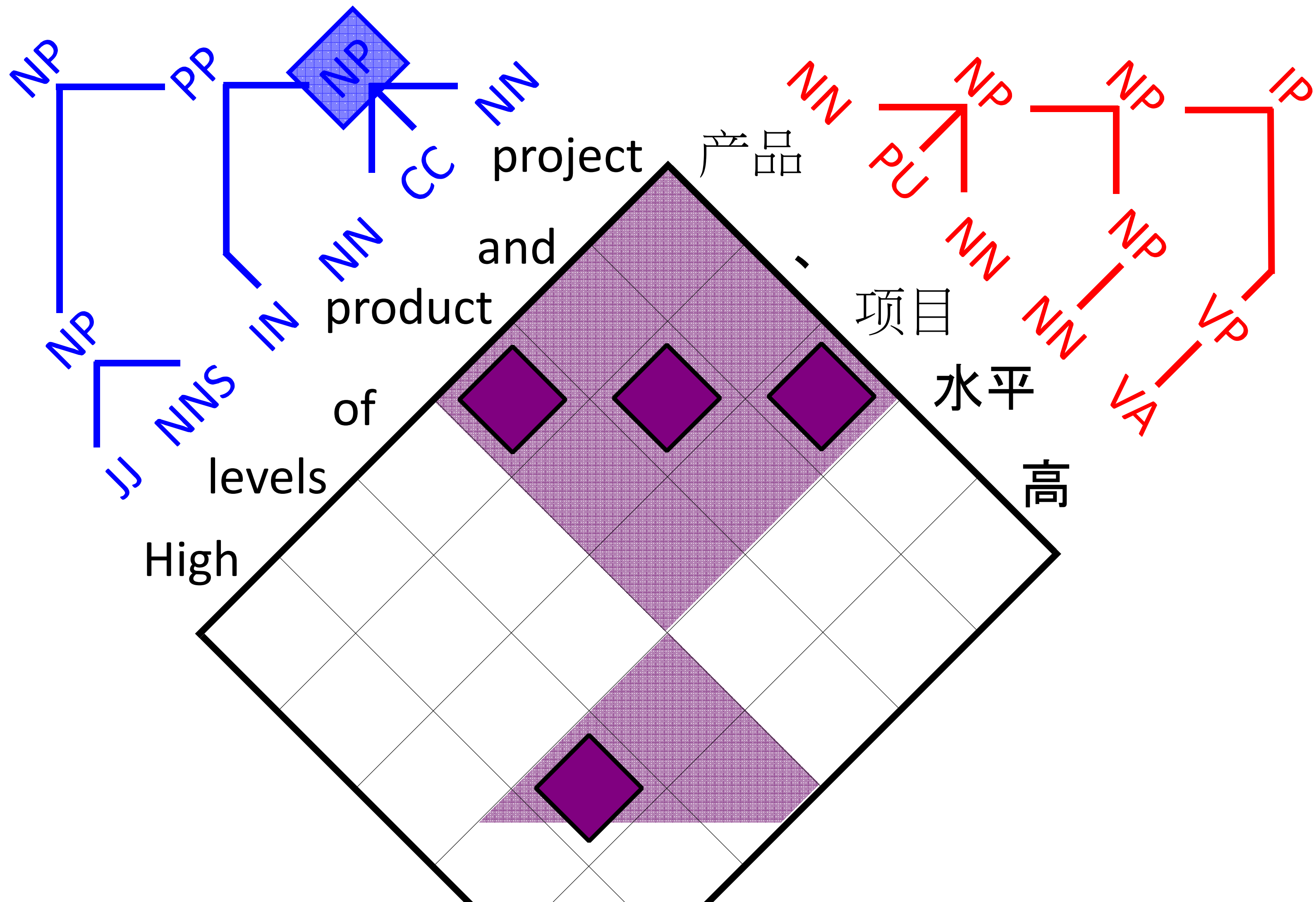
Joint Parsing and Alignment

Output: (t, a, t')



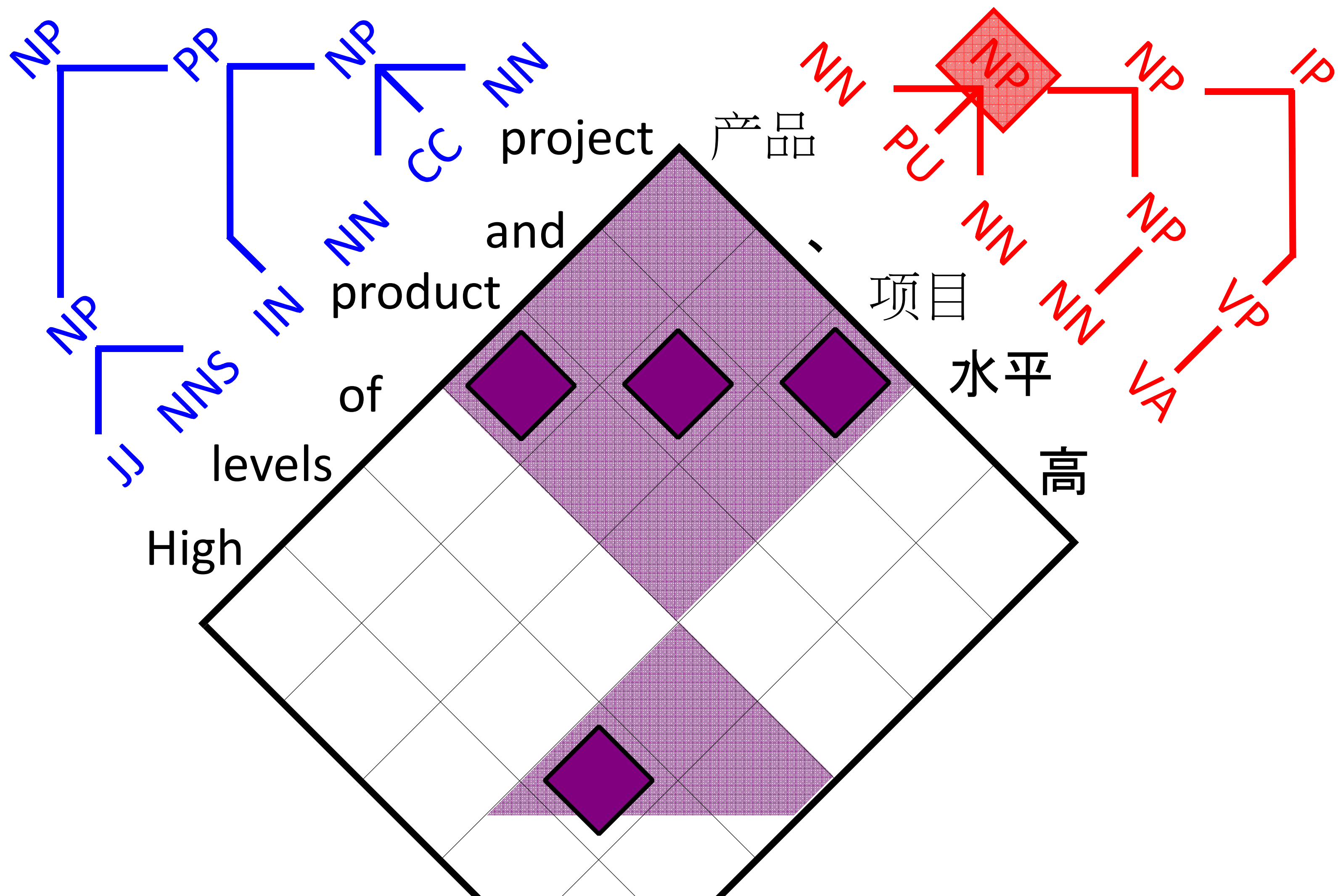
Joint Parsing and Alignment

Variables $n \in \{\text{true}, \text{false}\}$ $N_{_3\text{NP}_6} = \text{true}$



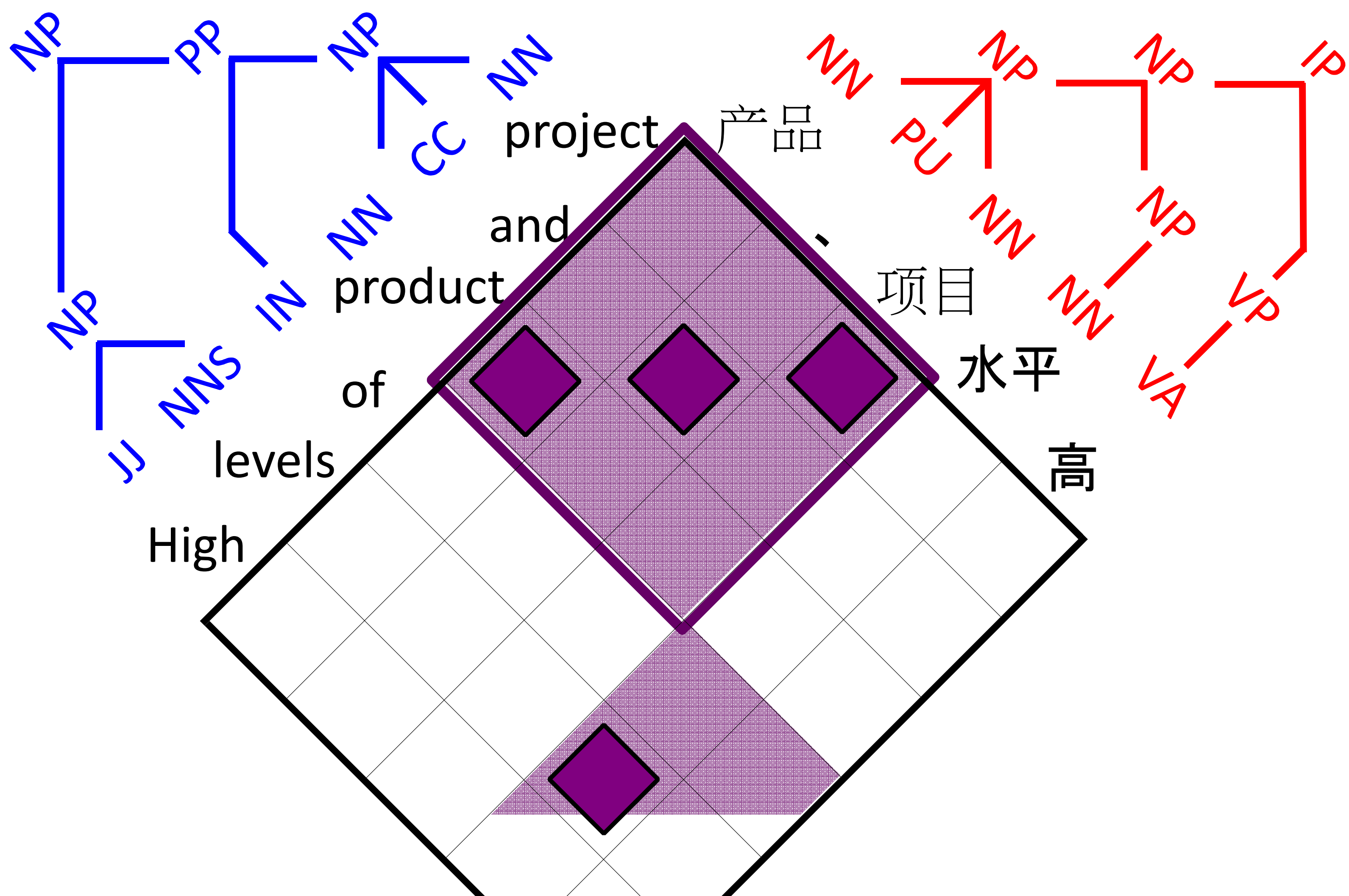
Joint Parsing and Alignment

Variables $\color{red}{n'} \in \{\text{true}, \text{false}\}$ $\color{red}{N'_{{}_0\mathrm{NP}_3}} = \text{true}$



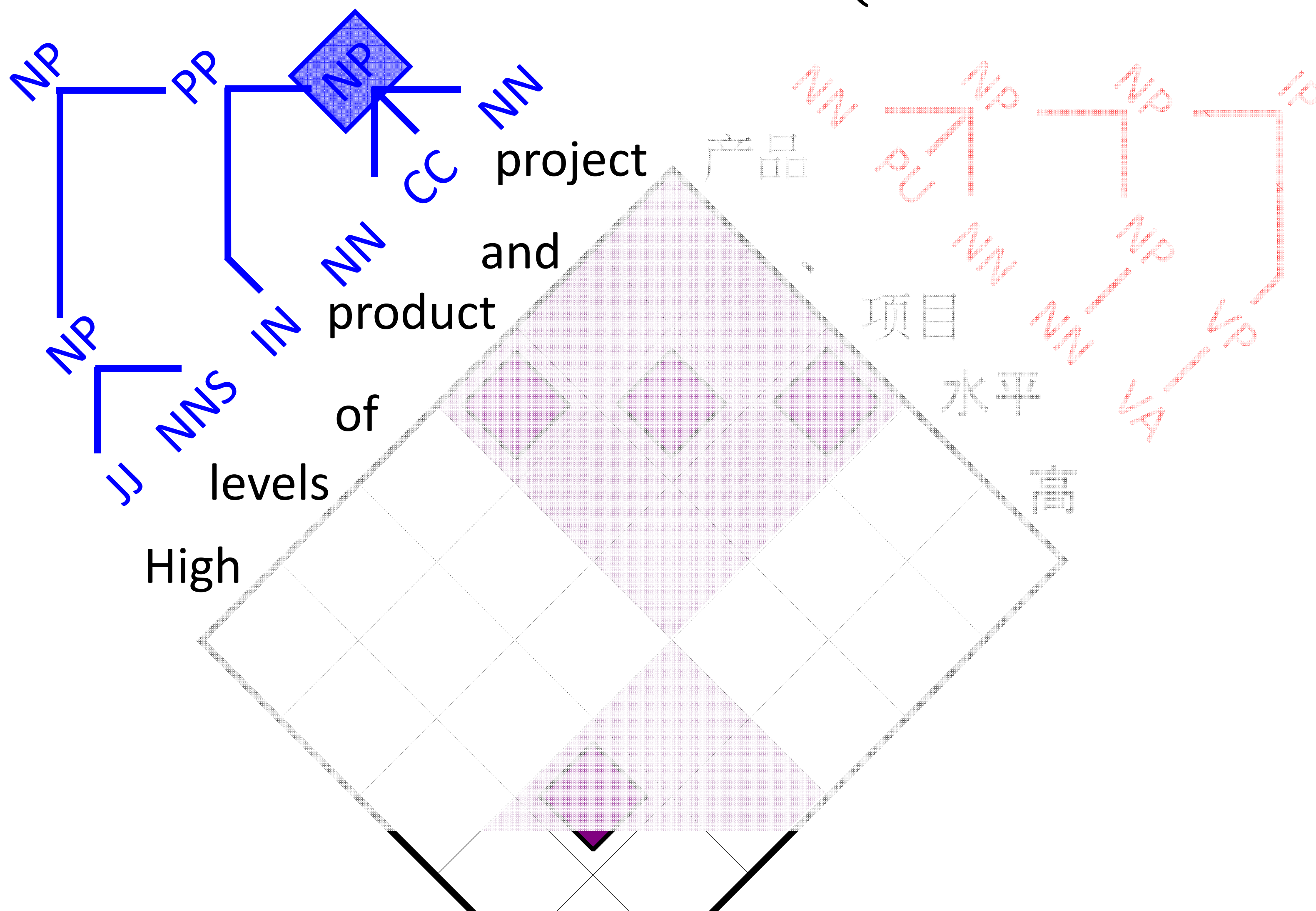
Joint Parsing and Alignment

Variables $b \in \{\text{true}, \text{false}\}$ $B_{36,03} = \text{true}$



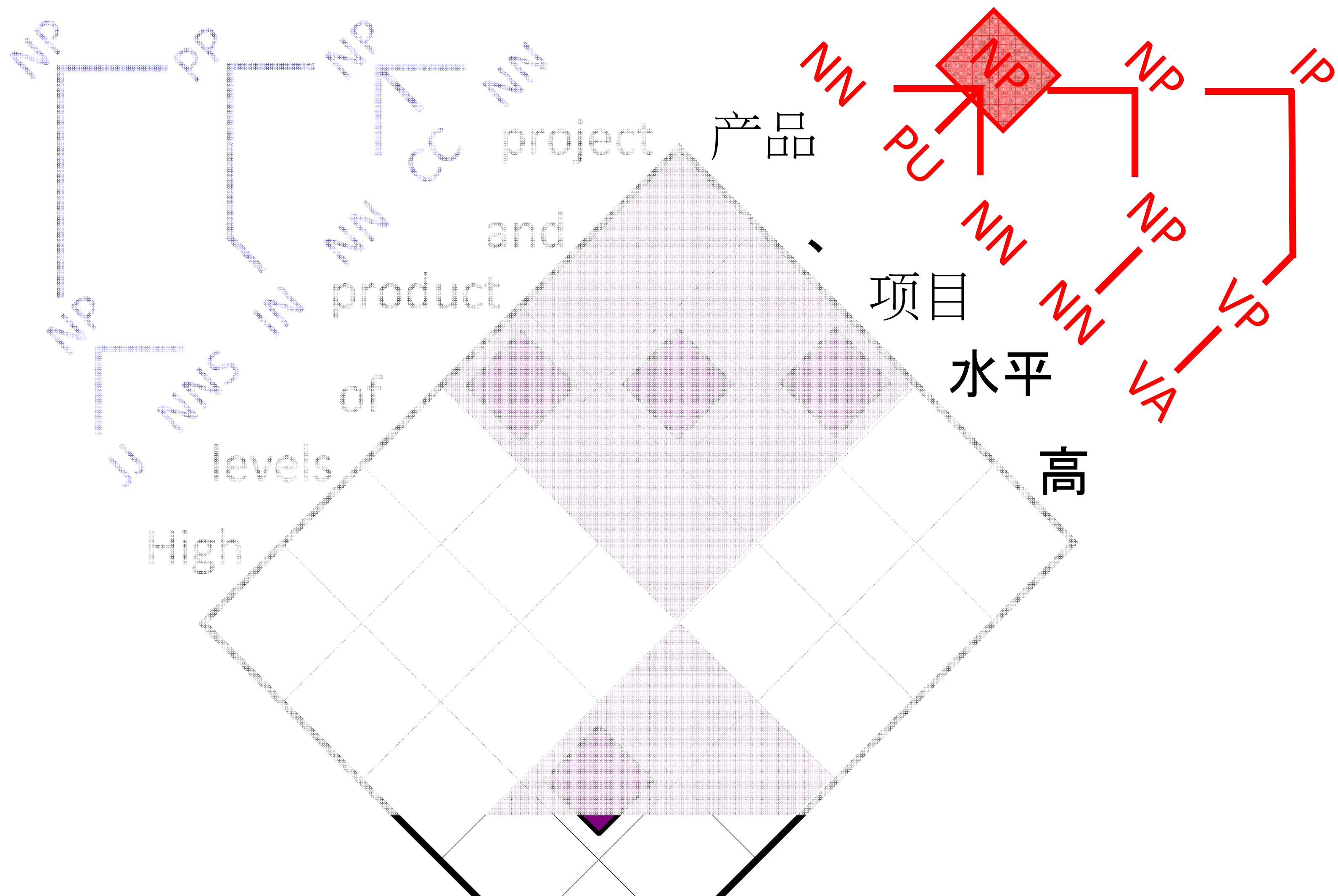
Joint Parsing and Alignment

Factors $\phi(n) = \exp(w^\top f_t(n))$ $\phi(t) = \begin{cases} 1 & t \text{ forms a tree} \\ 0 & \text{otherwise} \end{cases}$



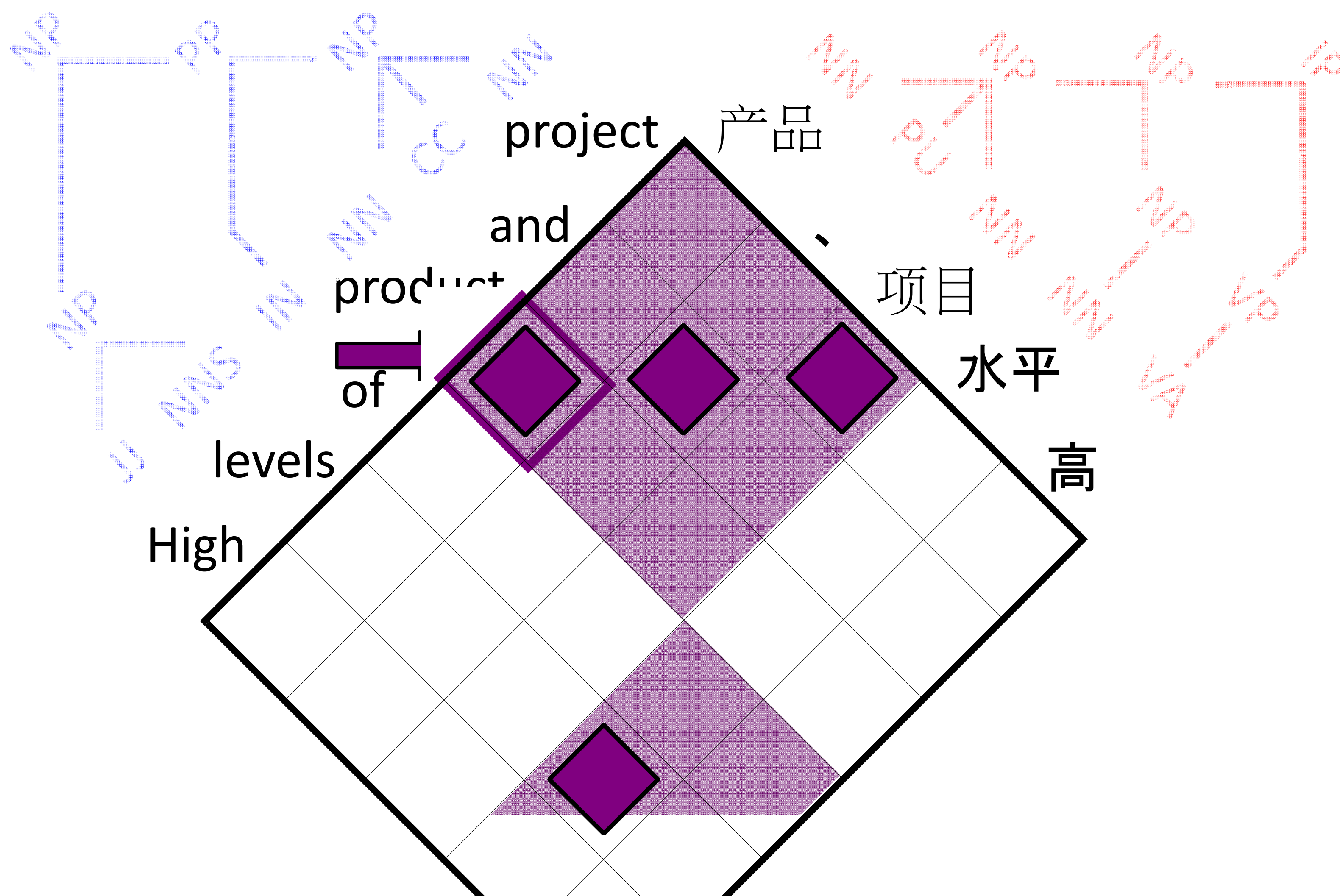
Joint Parsing and Alignment

Factors $\phi(n') = \exp(w^\top f_{t'}(n'))$ $\phi(t') = \begin{cases} 1 & t' \text{ forms a tree} \\ 0 & \text{otherwise} \end{cases}$



Joint Parsing and Alignment

Factors $\phi(b) = \exp(w^\top f_a(b))$ $\phi(a) = \begin{cases} 1 & a \text{ is an ITG derivation} \\ 0 & \text{otherwise} \end{cases}$

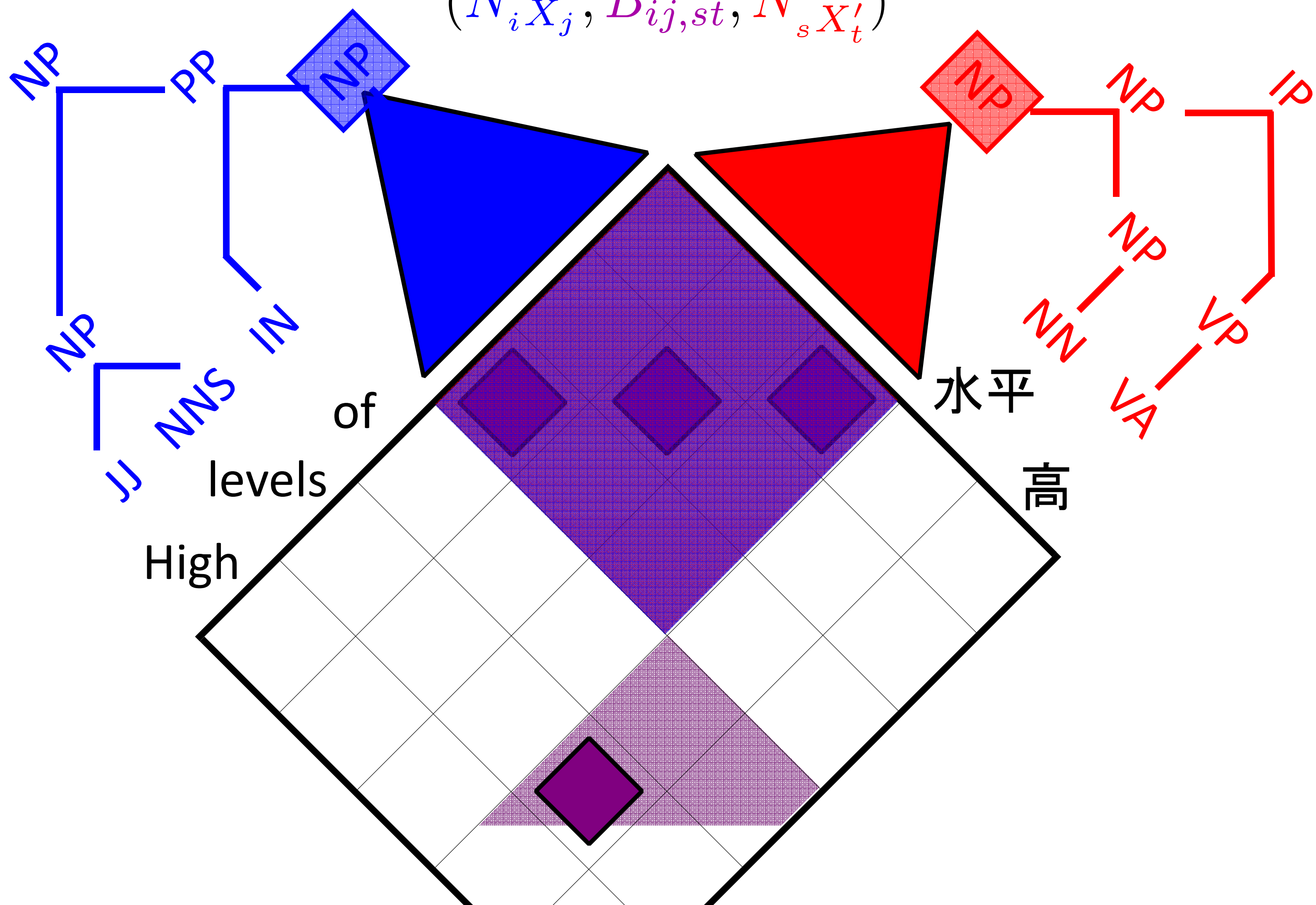


Joint Parsing and Alignment

Factors

$$\phi(n, b, n') = \exp(w^\top f_{tat'}(n, b, n'))$$

$$(N_{iX_j}, B_{ij,st}, N'_{sX'_t})$$



Notational Abuse

Subscript Omission:

$$f_t(n) = f_t(n_{{}_iX_j})$$

Shorthand:

$$n \in t \quad \Leftrightarrow \quad N_{{}_iX_j} = \text{true}$$

$$n \rhd b \lhd n' \quad \Leftrightarrow \quad n \in t \;\; \& \;\; b \in a \;\; \& \;\; n' \in t' \;\; \& \;\; (N_{{}_iX_j}, B_{ij,st}, N'_{{}_sX'_t}) \text{ match up}$$

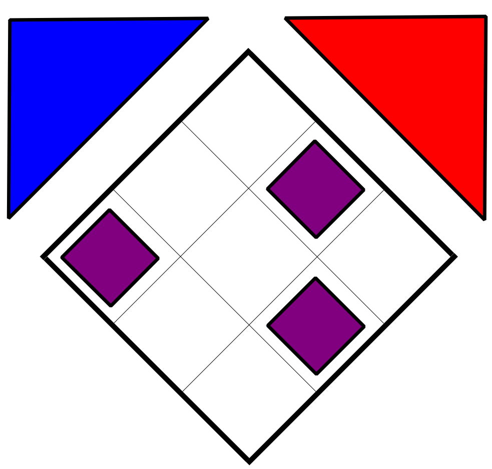
Skip Nonexistent Substructures:

$$n \notin t \quad \Rightarrow \quad f_t(n) = 0$$

Structural factors $\phi(t), \phi(a), \phi(t')$ are implicit

Model Form

$$P(t, a, t' | s, s') \propto \exp\left(\sum_{n \in t} w^\top f_t(n) + \sum_{b \in a} w^\top f_a(b) + \sum_{n' \in t'} w^\top f_{t'}(n') + \sum_{n \triangleright b \triangleleft n'} w^\top f_{tat'}(n, b, n') \right)$$



Training

Expected Feature Counts	Marginals
$\mathbb{E} f_t(n)$	$P(n \in t \| s, s')$
$\mathbb{E} f_a(b)$	$P(b \in a \| s, s')$
$\mathbb{E} f_{t'}(n')$	$P(n' \in t' \| s, s')$
$\mathbb{E} f_{tat'}(n, b, n')$	$P(n \triangleright b \triangleleft n' \| s, s')$

Structured Mean Field Approximation

$$P(t, a, t' | s, s') \propto \exp\left(\sum_{n \in t} w^\top f_t(n) + \sum_{b \in a} w^\top f_a(b) + \sum_{n' \in t'} w^\top f_{t'}(n') + \sum_{n \triangleright b \triangleleft n'} w^\top f_{tat'}(n, b, n')\right)$$

$$\approx q(t)q(a)q(t')$$

Approximate Component Scores

Monolingual parser:

Score for $n = w^\top f_t(n)$

If we knew (a, t'):

Score for $n = w^\top f_t(n) + w^\top f_{tat'}(n, b, n')$

To compute $q(t)$:

Score for $n = w^\top f_t(n) + w^\top \mathbb{E}_{q(a,t')} f_{tat'}(n, b, n')$

Expected Feature Counts

For fixed $n_{{}_iX_j}$:

$$\mathbb{E}_{q(a,t')} f_{tat'}(n, b, n')$$

$$= \sum_{{}_sX'_t} P_q(n_{{}_iX_j} \triangleright b_{ij,st} \triangleleft n'_{{}_sX'_t}) f_{tat'}(n, b, n')$$

$$= \sum_{{}_sX'_t} q(b_{ij,st}) q(n'_{{}_sX'_t}) f_{tat'}(n, b, n')$$

Marginals computed with bitext inside-outside

Marginals computed with inside-outside

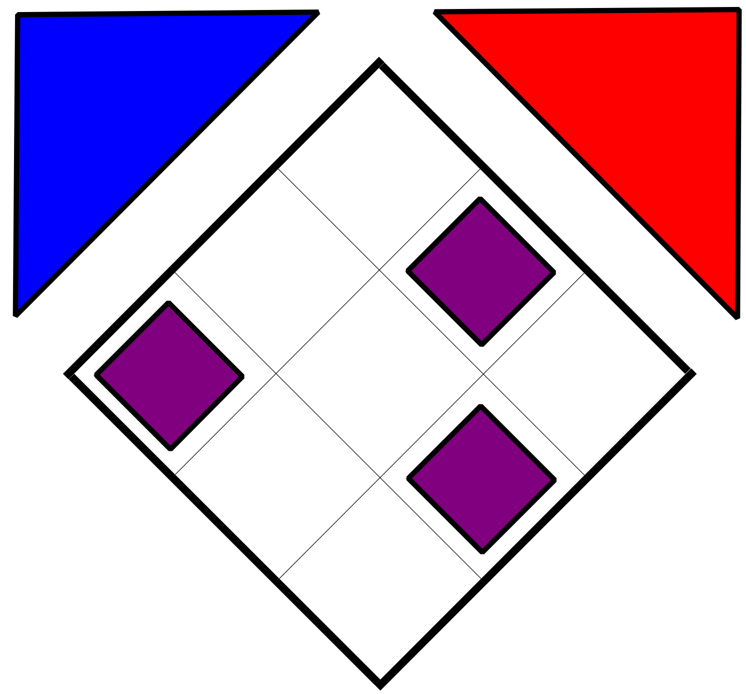
Inference Procedure

Initialize:

$$q(t) \propto \exp\left(\sum_{n \in t} w^\top f_t(n)\right)$$

$$q(a) \propto \exp\left(\sum_{b \in a} w^\top f_a(b)\right)$$

$$q(t') \propto \exp\left(\sum_{n' \in t'} w^\top f_{t'}(n')\right)$$



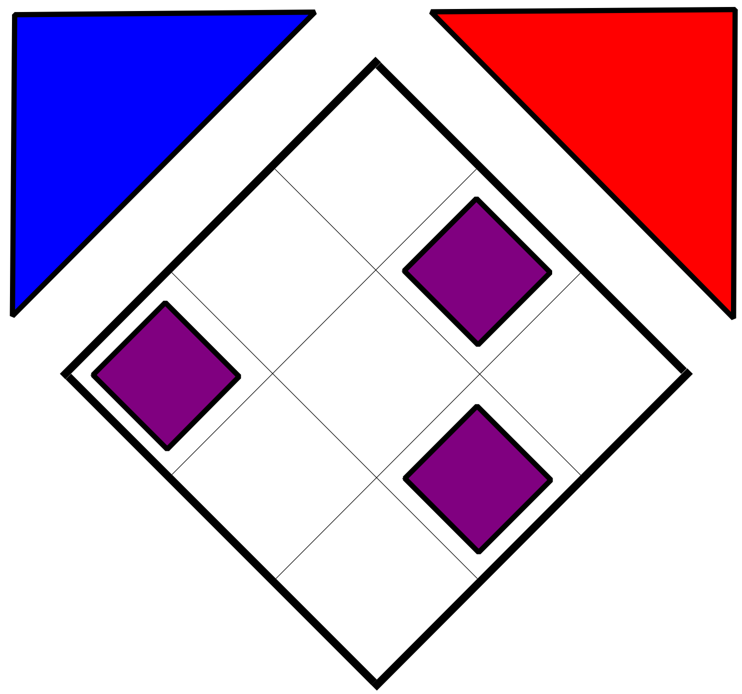
Inference Procedure

Iterate marginal updates:

$$q(n)$$

$$q(b)$$

$$q(n')$$



...until convergence!

Approximate Marginals

$$P(n \in t | s, s') \approx q(n)$$

$$P(b \in a | s, s') \approx q(b)$$

$$P(n' \in t' | s, s') \approx q(n')$$

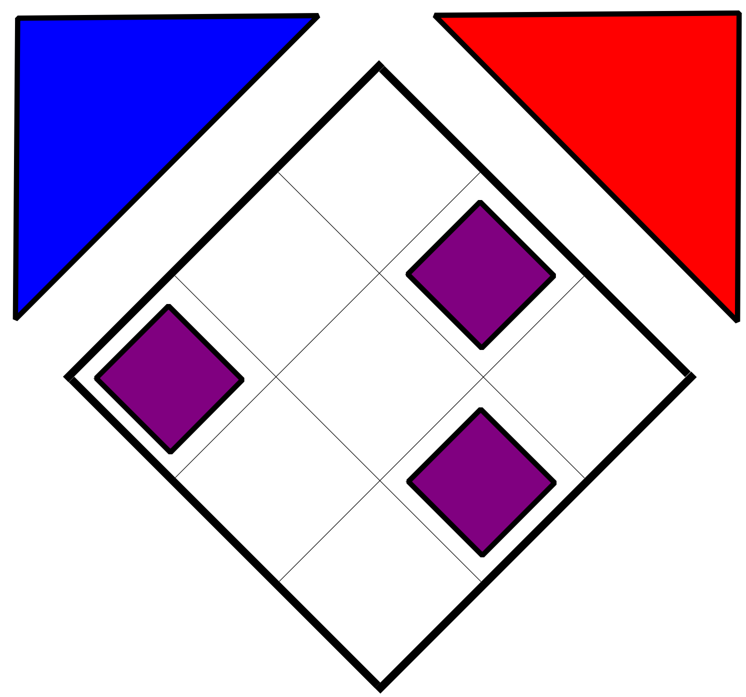
$$P(n \rhd b \lhd n' | s, s') \approx q(n)q(b)q(n')$$

Decoding

$$\hat{t} = \underset{t}{\operatorname{argmax}}\, q(t)$$

$$\hat{a} = \underset{a}{\operatorname{argmax}}\, q(a)$$

$$\hat{t'} = \underset{t'}{\operatorname{argmax}}\, q(t')$$



(Minimum Risk)

Structured Mean Field Summary

- Split the model into pieces you have dynamic programs for
- Substitute expected feature counts for actual counts in cross-component factors
- Iterate computing marginals until convergence

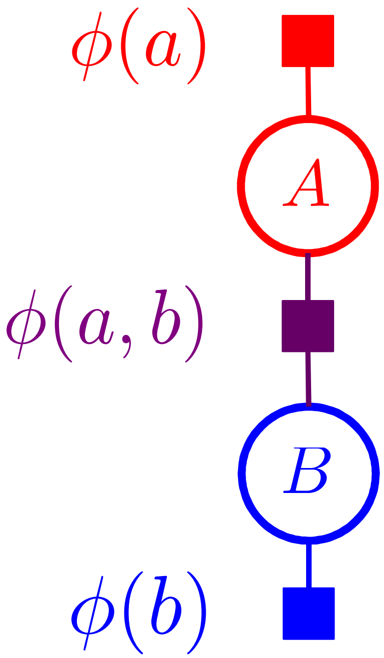
Structured Mean Field Tips

- Try to make sure cross-component features are products of indicators
- You don't have to run all the way to convergence; marginals are usually pretty good after just a few rounds
- Recompute marginals for fast components more frequently than for slow ones
  - e.g. For joint parsing and alignment, the two monolingual tree marginals ($O(n^3)$) were updated until convergence between each update of the ITG marginals ($O(n^6)$)

Break Time!

Part 4: Belief Propagation

Belief Propagation



Wanted: $P(a|x), P(b|x)$

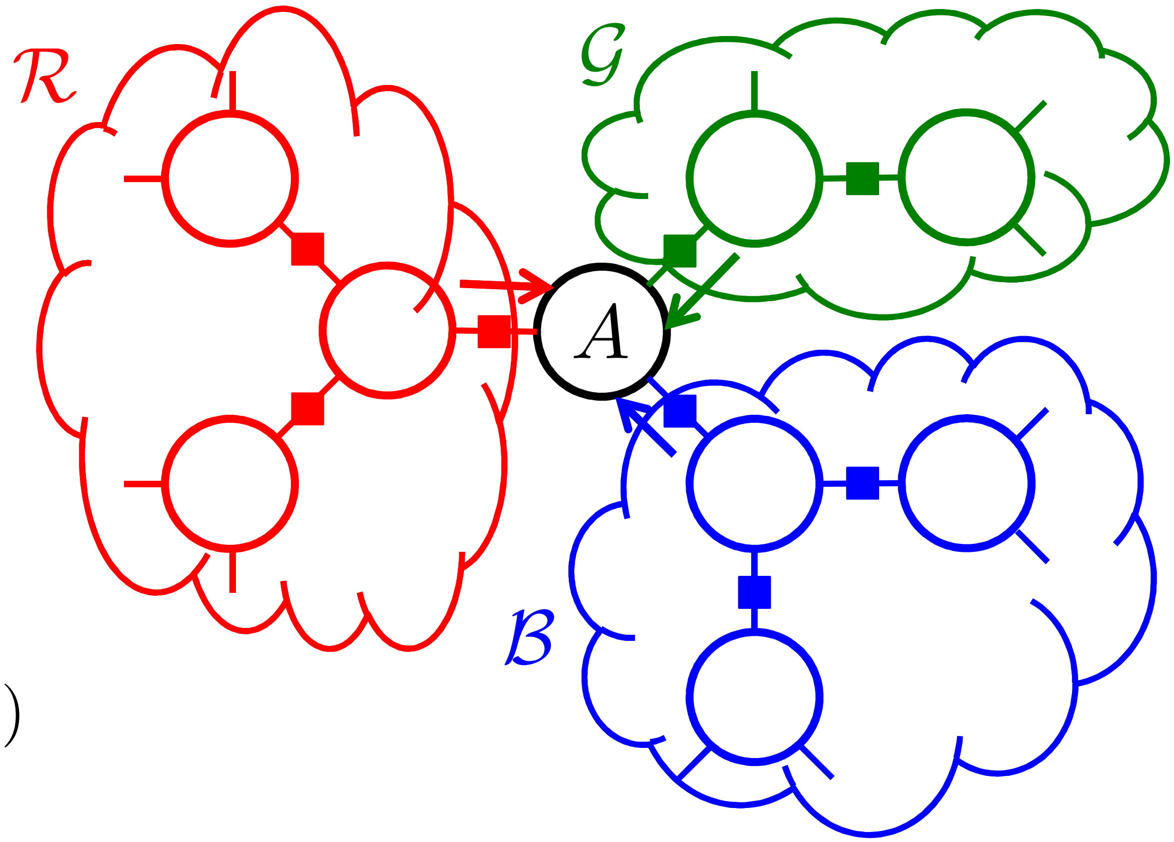
Idea: pretend graph is a tree

Key objects:

- Beliefs (marginals)
- Messages

Belief Propagation Intro

Assume we have a tree



$$P(a|x) \propto \sum_{y \backslash \{a\}} score(y)$$

$$= \left(\sum_{\mathcal{R}} score(\mathcal{R}, a) \right) \left(\sum_{\mathcal{G}} score(\mathcal{G}, a) \right) \left(\sum_{\mathcal{B}} score(\mathcal{B}, a) \right)$$

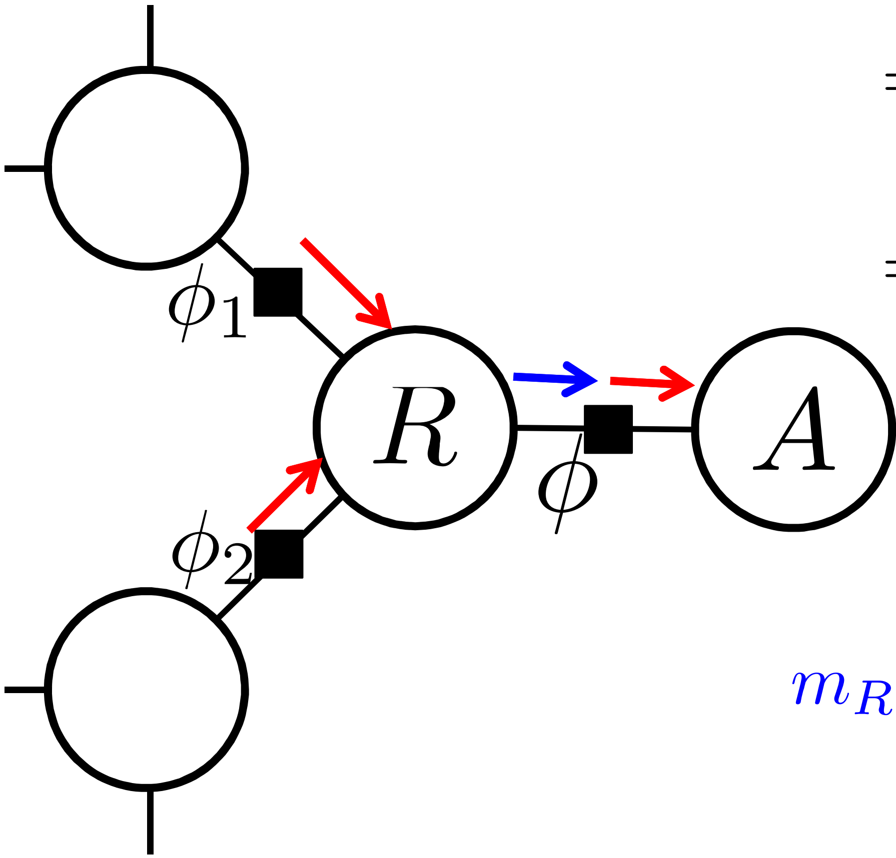
$$= m_{\mathcal{R}\to A}(a) \cdot m_{\mathcal{G}\to A}(a) \cdot m_{\mathcal{B}\to A}(a)$$

Belief Propagation Intro

$$m_{\phi\to A}(a) = \sum_{\mathcal{R}} score(\mathcal{R}, a)$$

$$= \sum_{r} \phi(r,a) m_{\phi_1\to R}(r) m_{\phi_2\to R}(r)$$

$$= \sum_{r} \phi(r,a) m_{R\to\phi}(r)$$

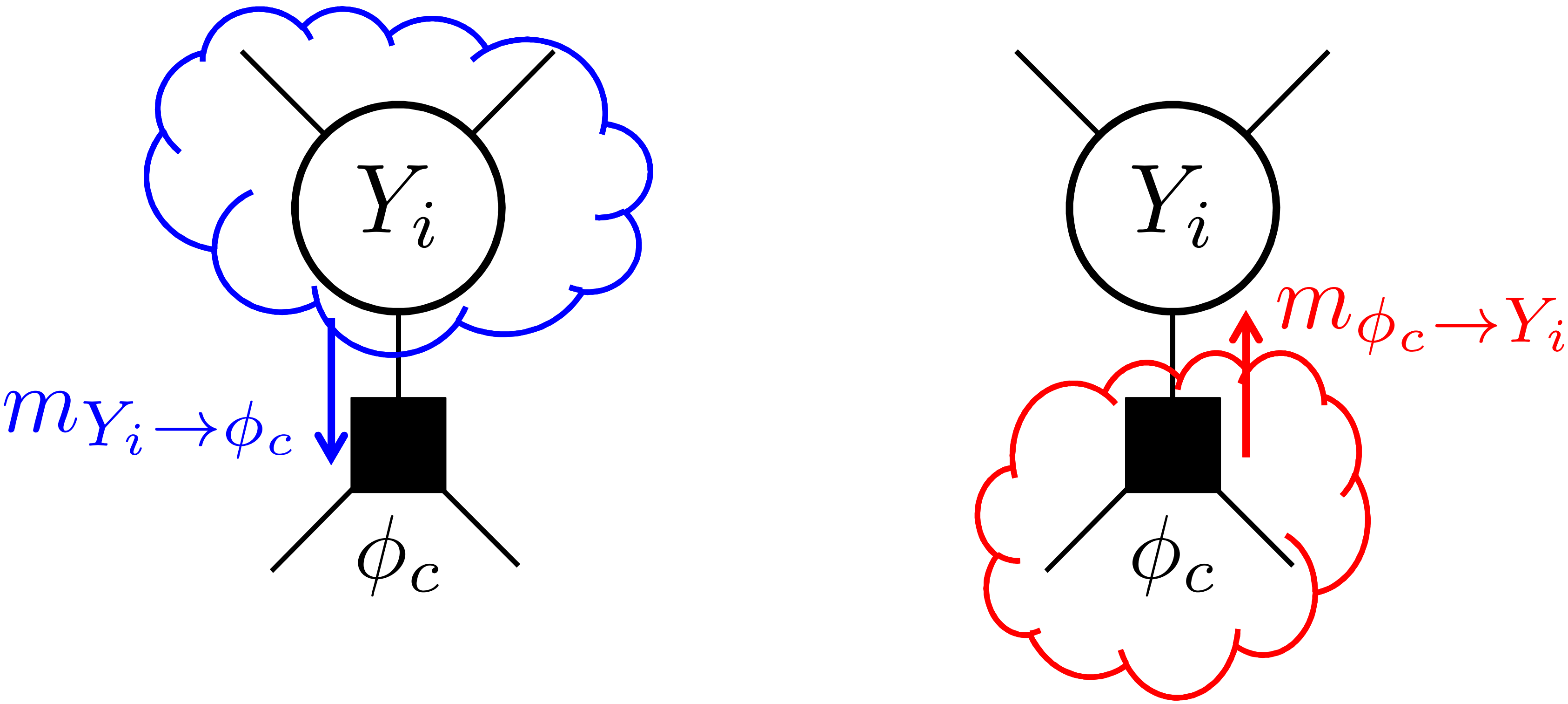


$$m_{R\to\phi}(r) = m_{\phi_1\to R}(r) m_{\phi_2\to R}(r)$$

Messages

Variable to Factor

Factor to Variable

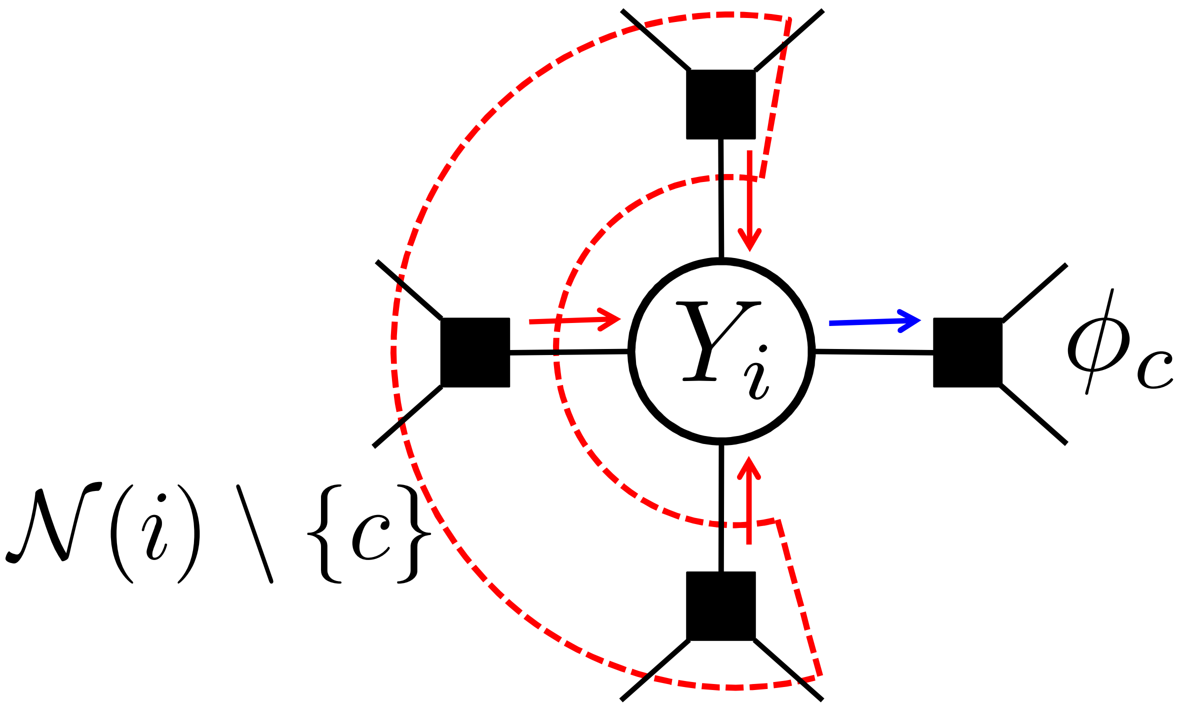


Both take form of "distribution" over Y_i

Messages General Form

- Messages from variables to factors:

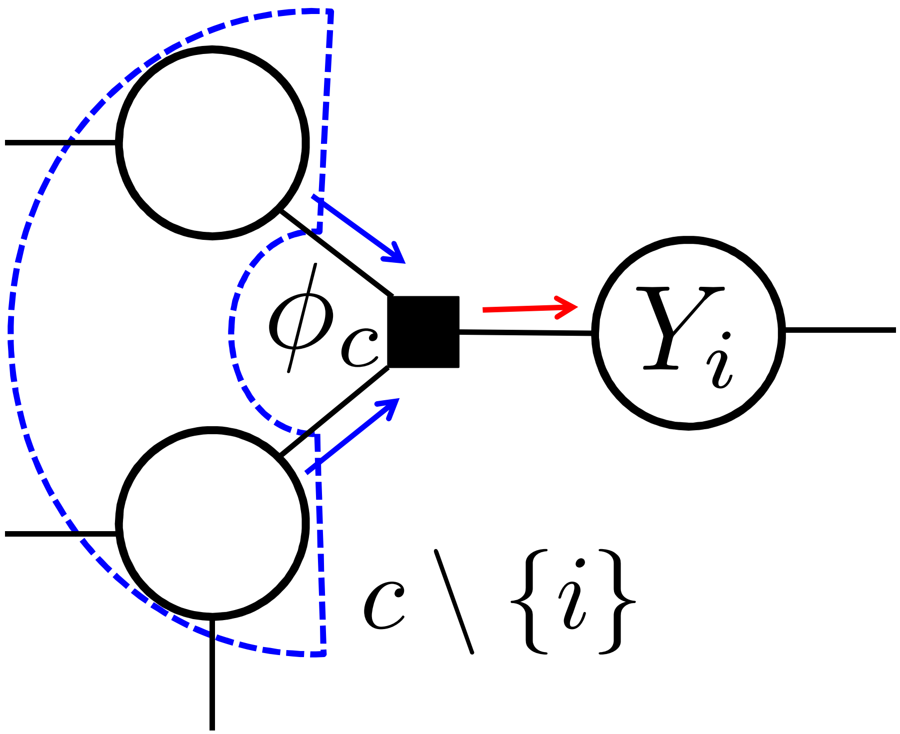
$$m_{Y_i \to \phi_c}(y_i) \propto \prod_{c' \in \mathcal{N}(i) \setminus \{c\}} m_{\phi_{c'} \to Y_i}(y_i)$$



Messages General Form

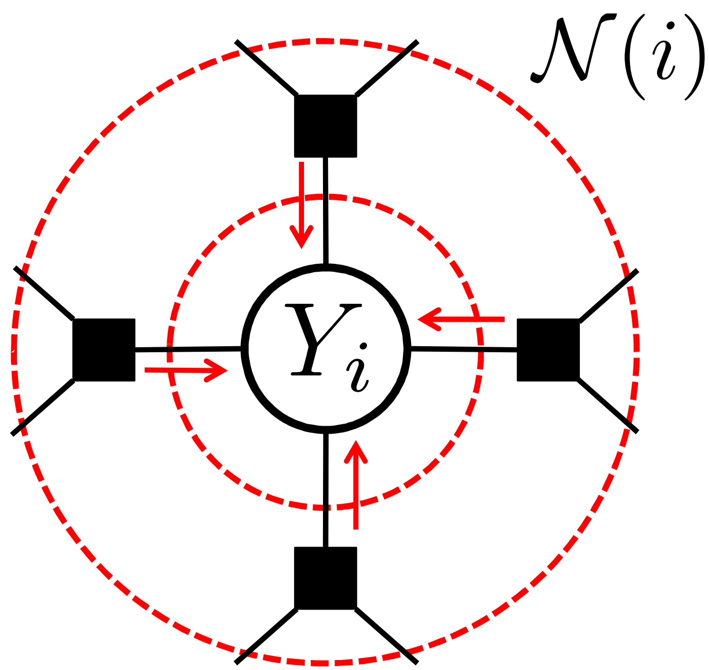
- Messages from factors to variables:

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_{c \setminus \{i\}}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



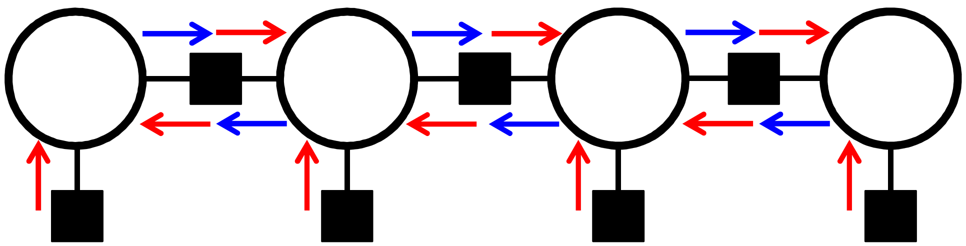
Marginal Beliefs

$$b_{Y_i}(y_i) \propto \prod_{c \in \mathcal{N}(i)} m_{\phi_c \to Y_i}(y_i)$$



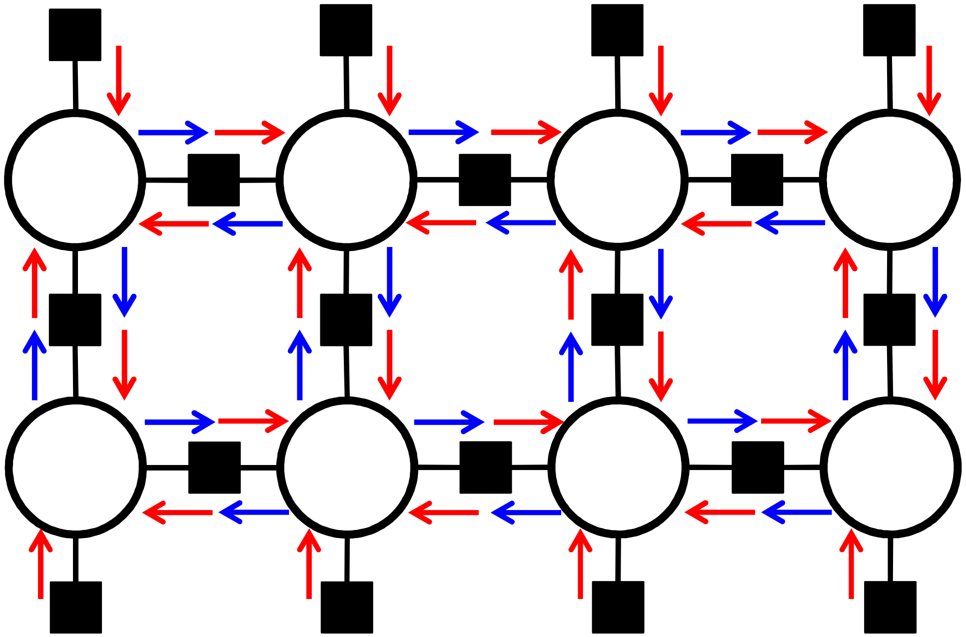
Belief Propagation on Tree-Structured Graphs

- If the factor graph has no cycles, BP is exact
  - Can always order message computations



- After one pass, marginal beliefs are correct

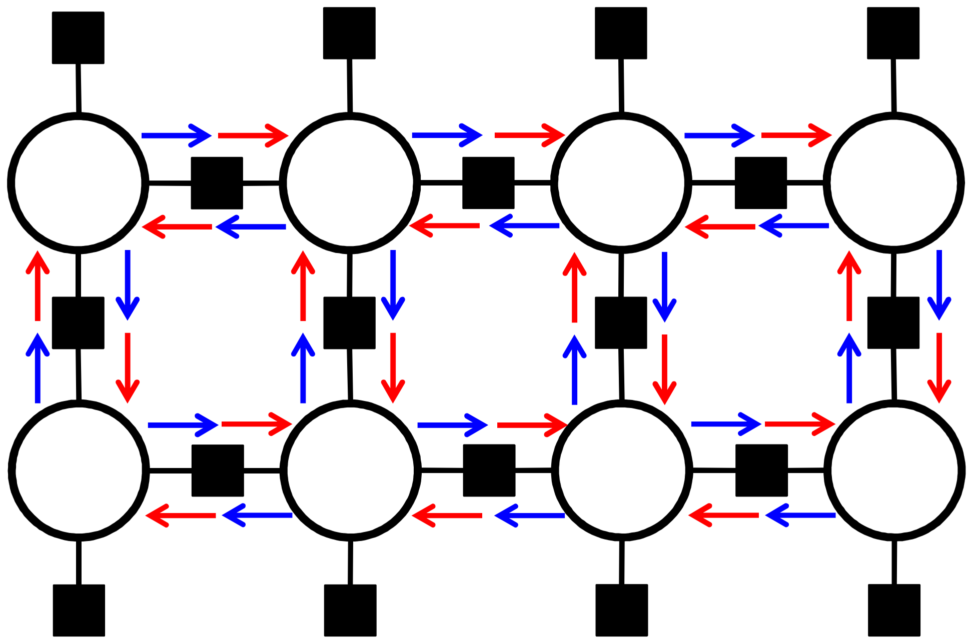
“Loopy” Belief Propagation



Problem: we no longer have a tree

Solution: ignore problem

“Loopy” Belief Propagation



Just start passing messages anyway!

Belief Propagation Q&A

- Are the marginals guaranteed to converge to the right thing, like in sampling?

  No

- Well, is the algorithm at least guaranteed to converge to something, like mean field?

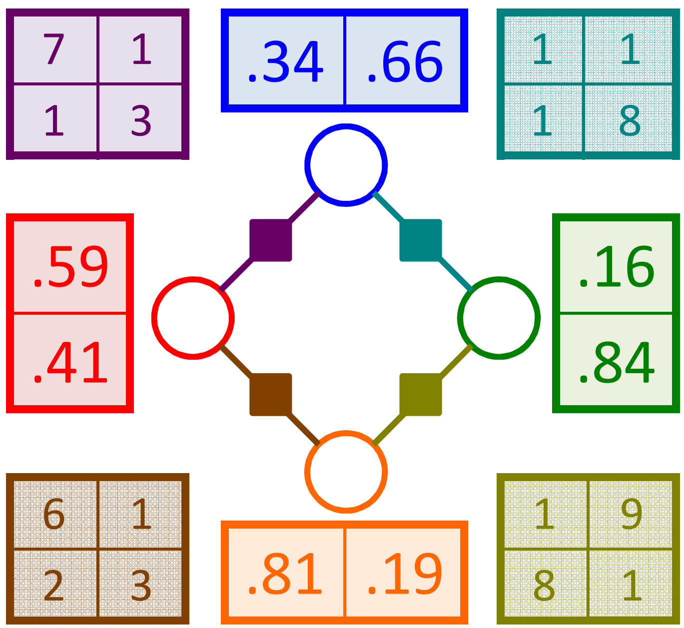
  No

- Will everything often work out more or less OK in practice?

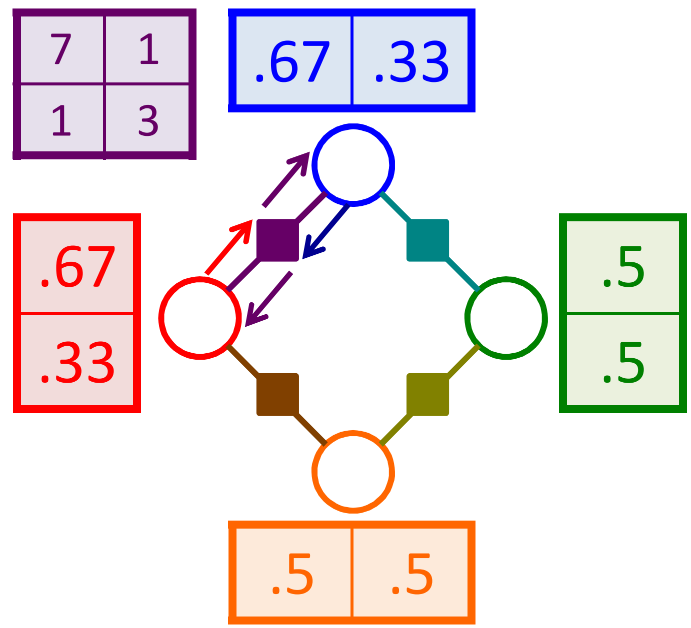
  Maybe

Belief Propagation Example

Exact



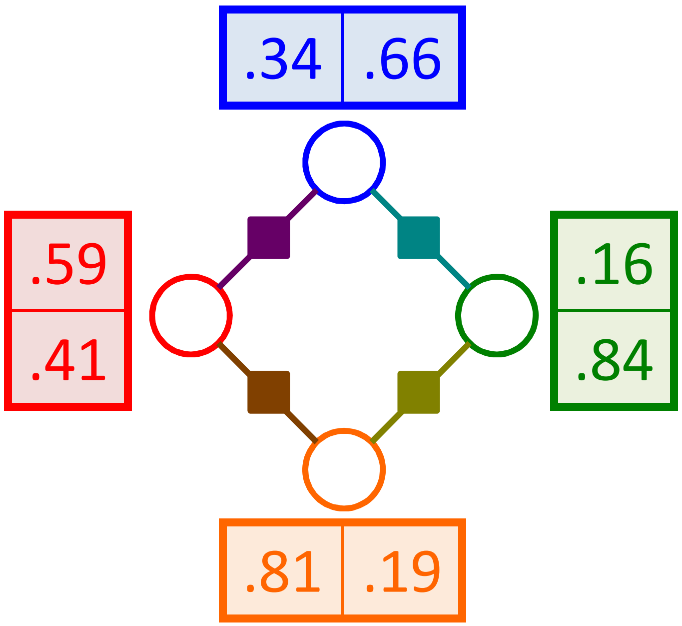
BP

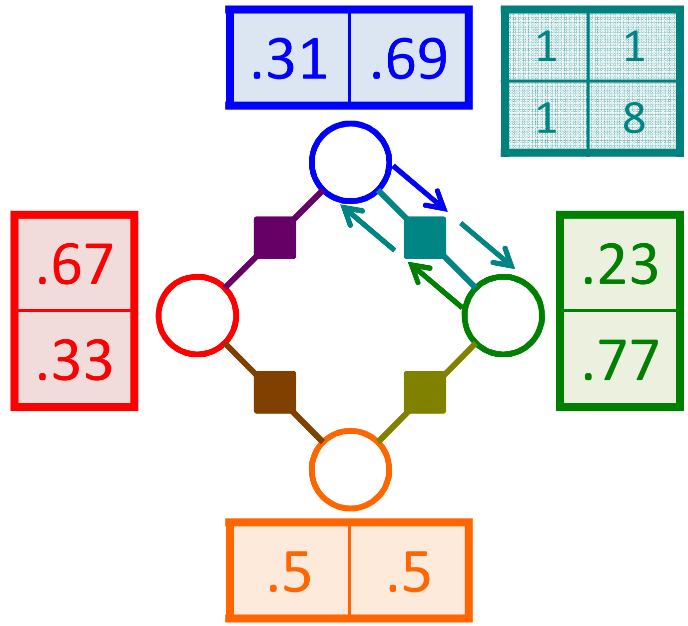


Belief Propagation Example

Exact

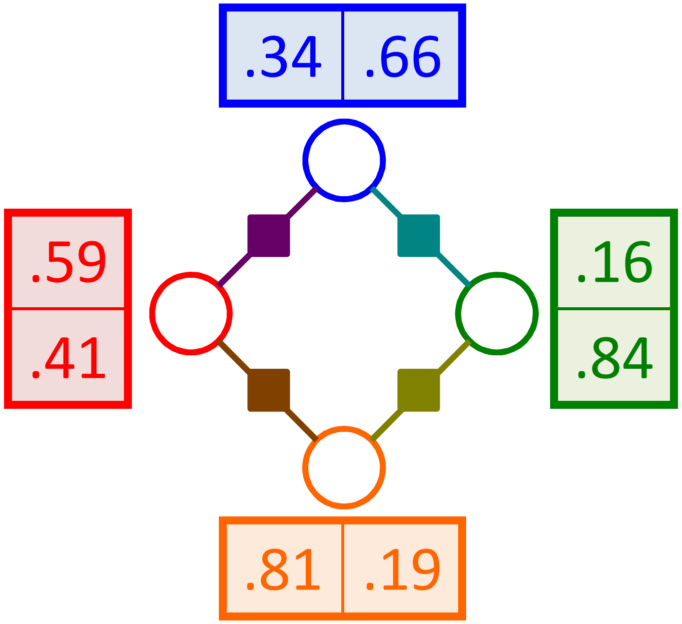
BP



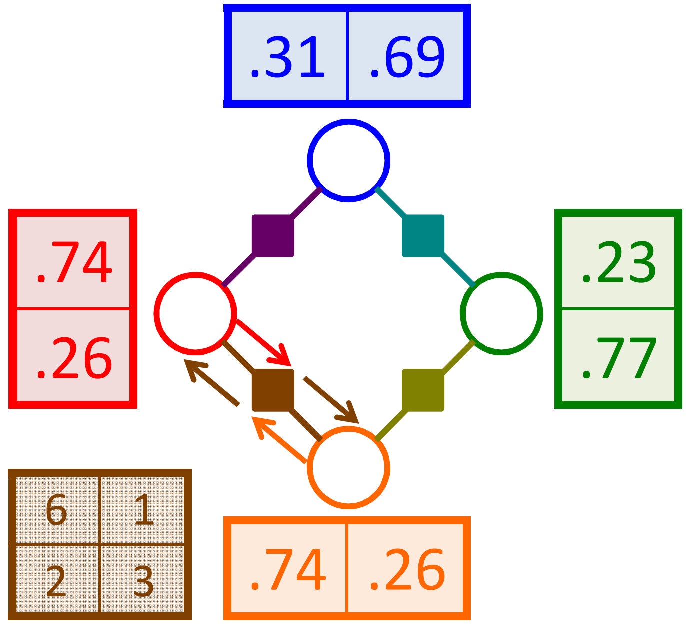


Belief Propagation Example

Exact

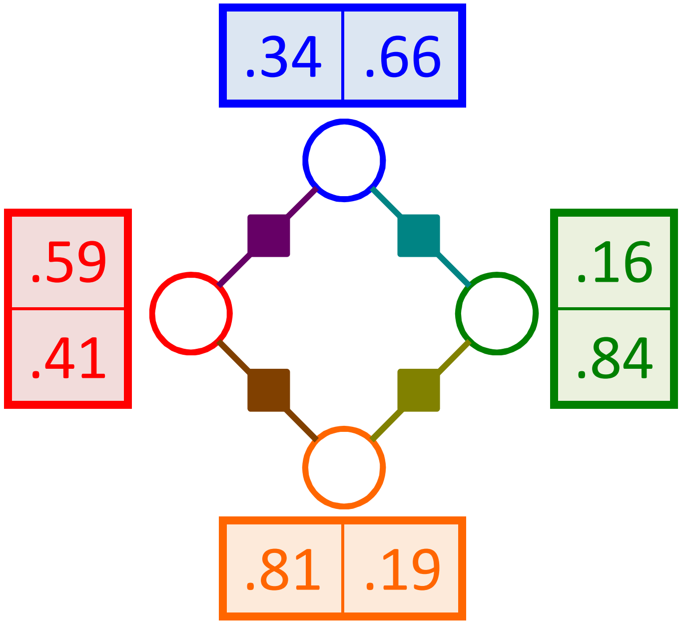


BP



Belief Propagation Example

Exact



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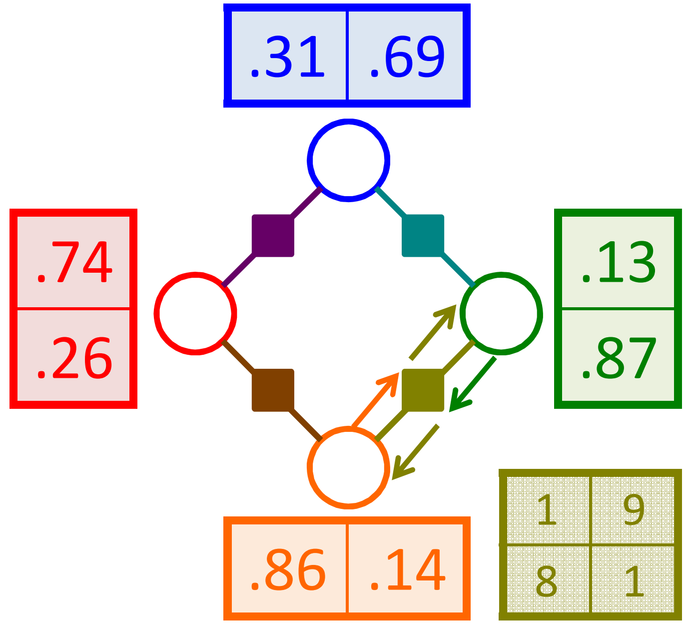
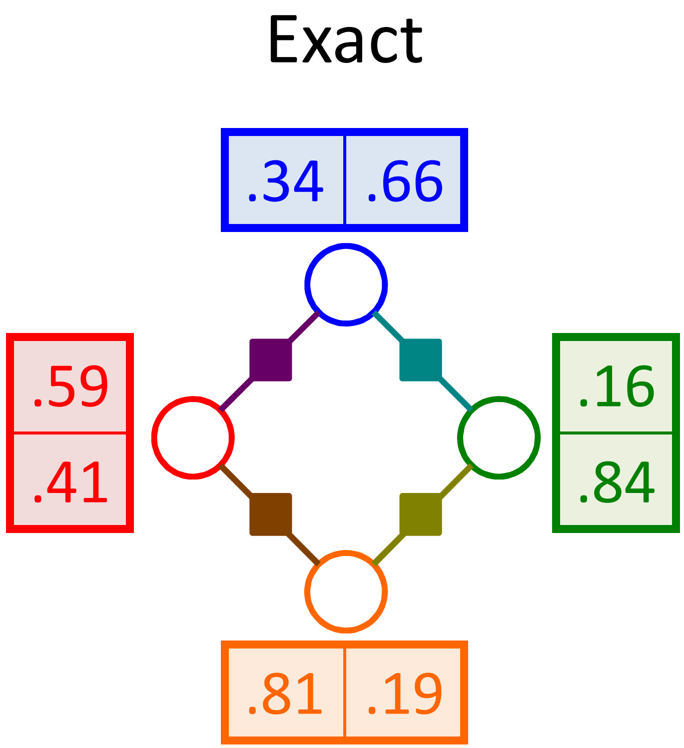
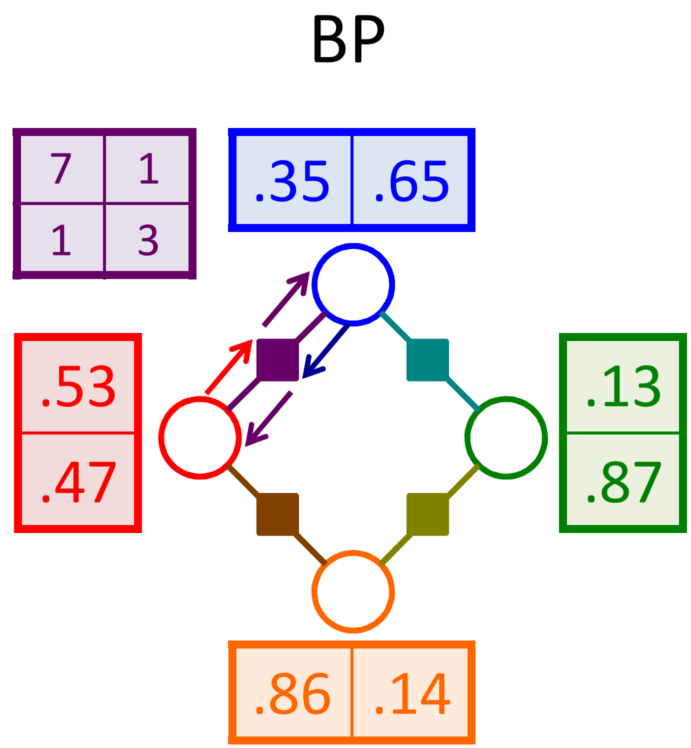


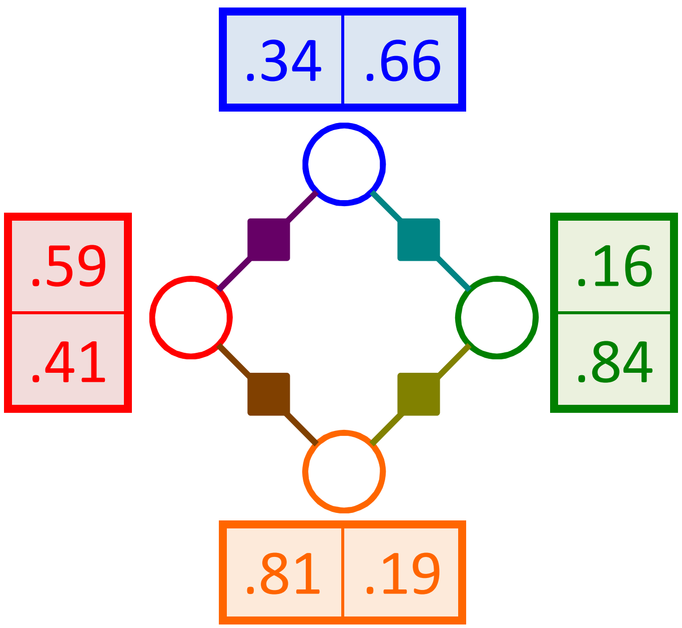
Belief Propagation Example



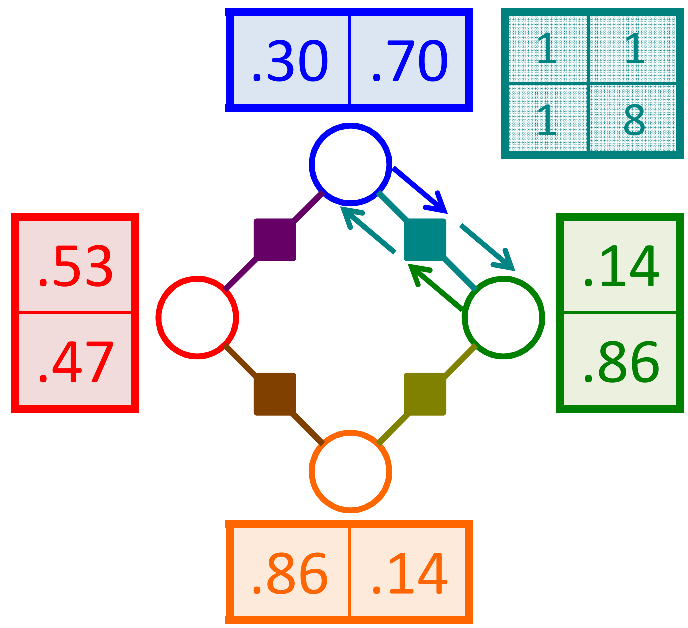


Belief Propagation Example

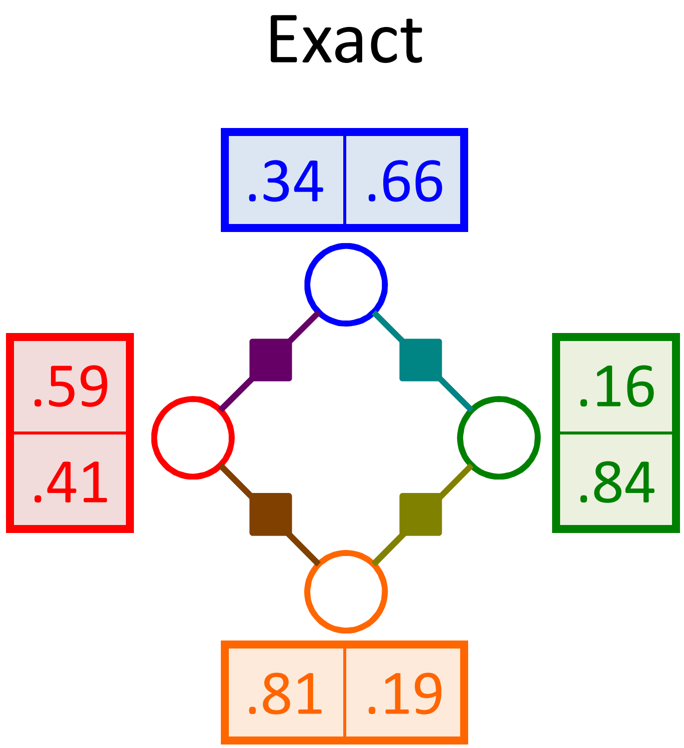
Exact

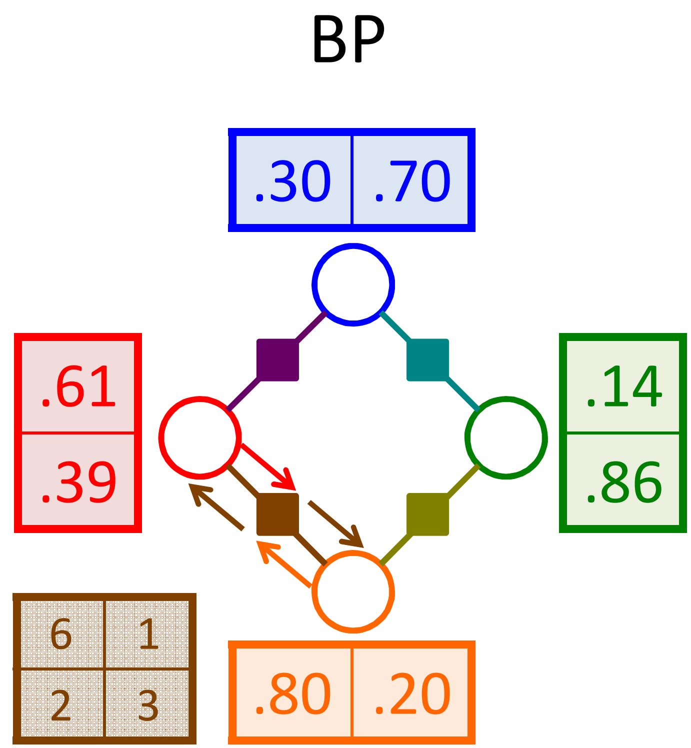


BP



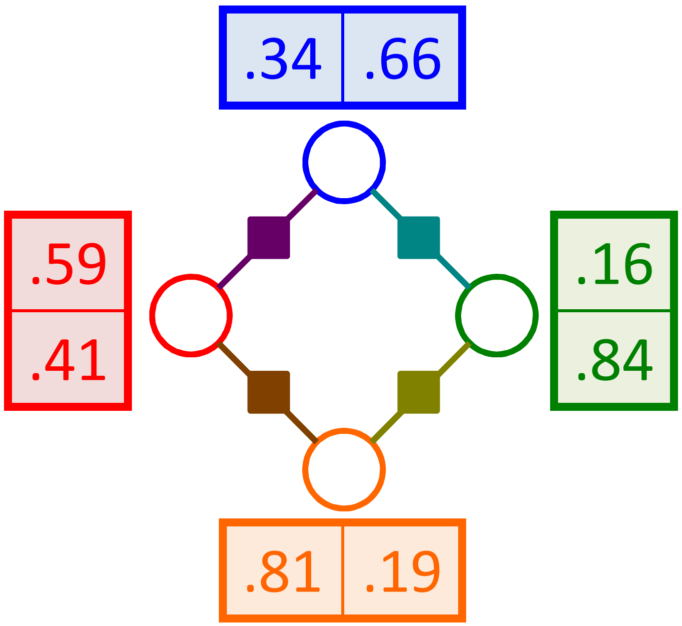
Belief Propagation Example



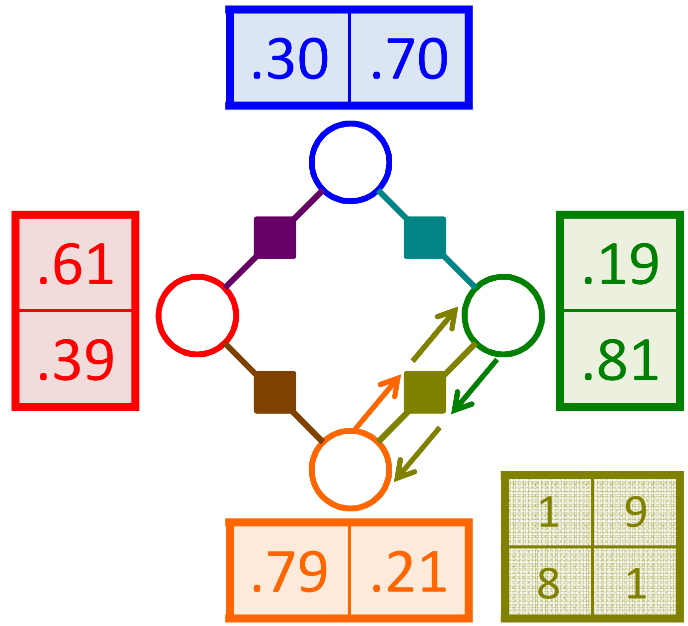


Belief Propagation Example

Exact

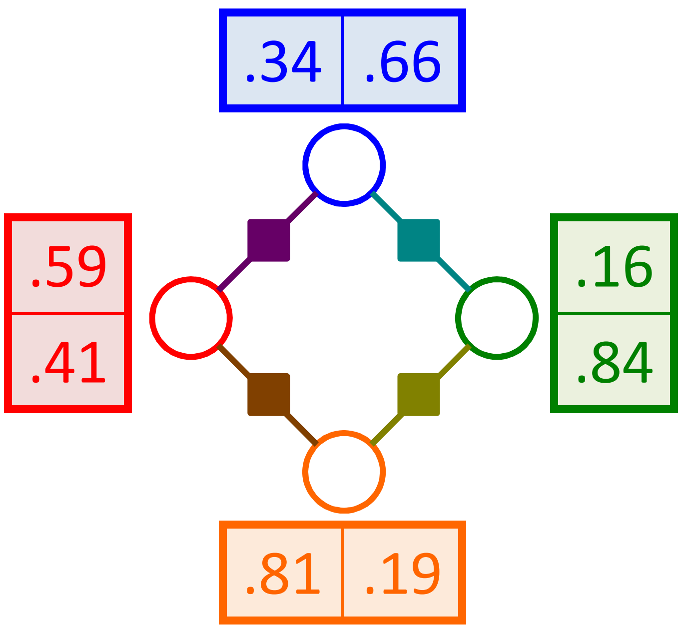


BP

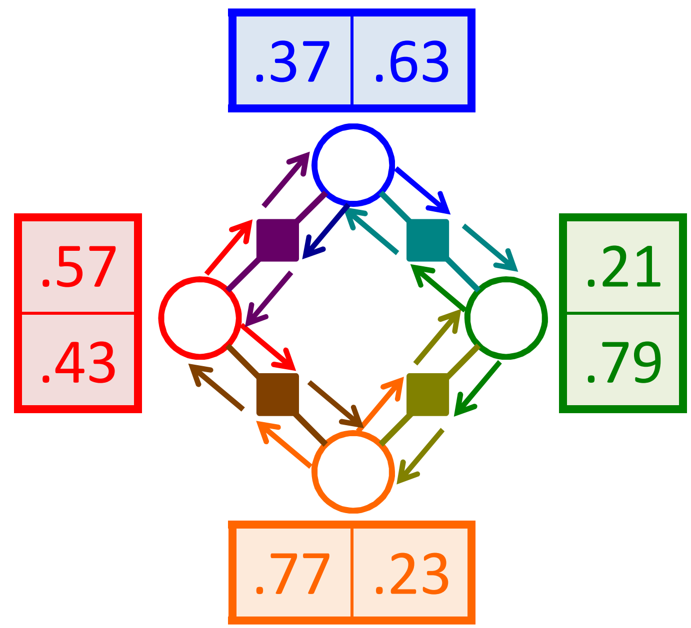


Belief Propagation Example

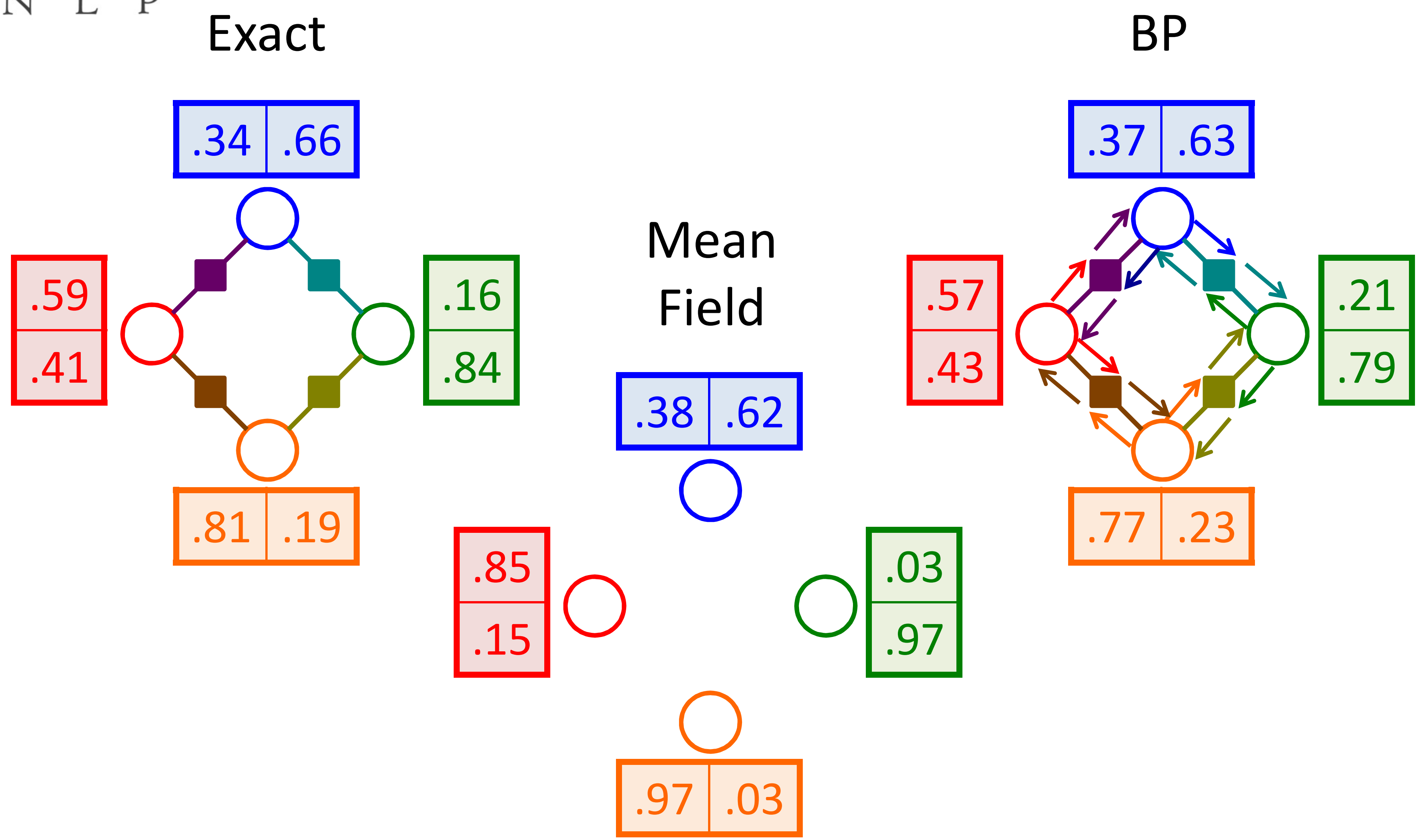
Exact



BP

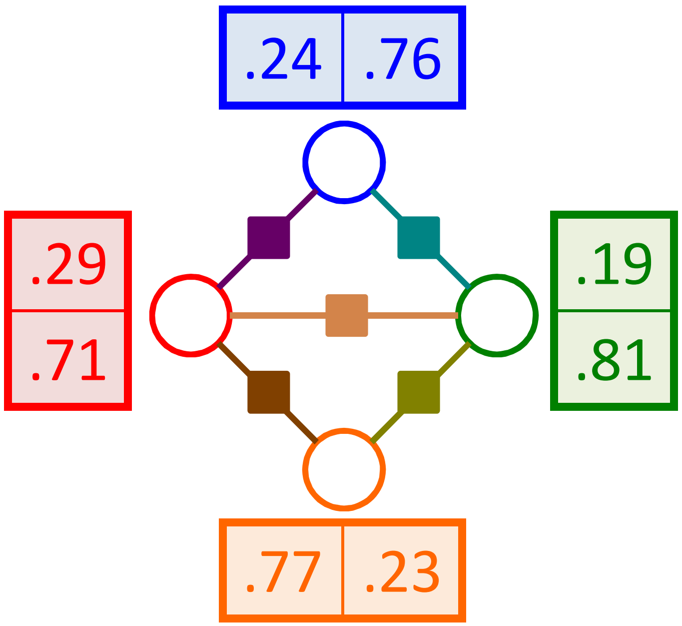


Belief Propagation Example



Belief Propagation Example

Exact



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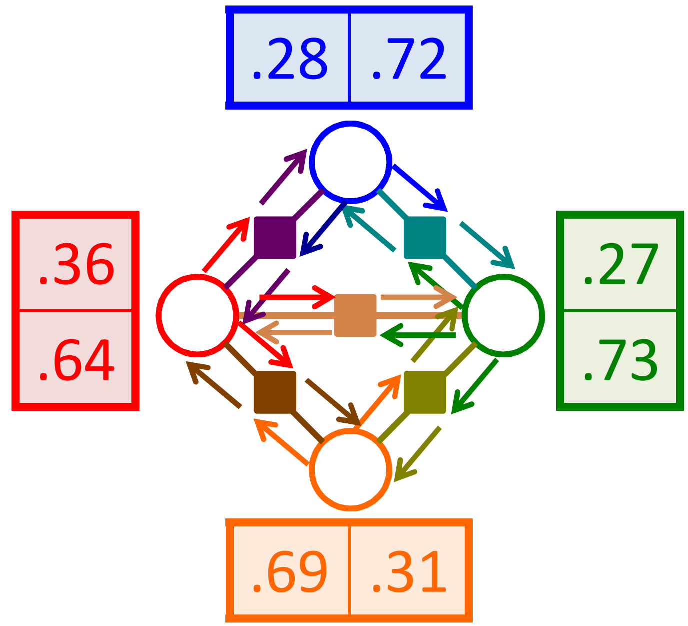
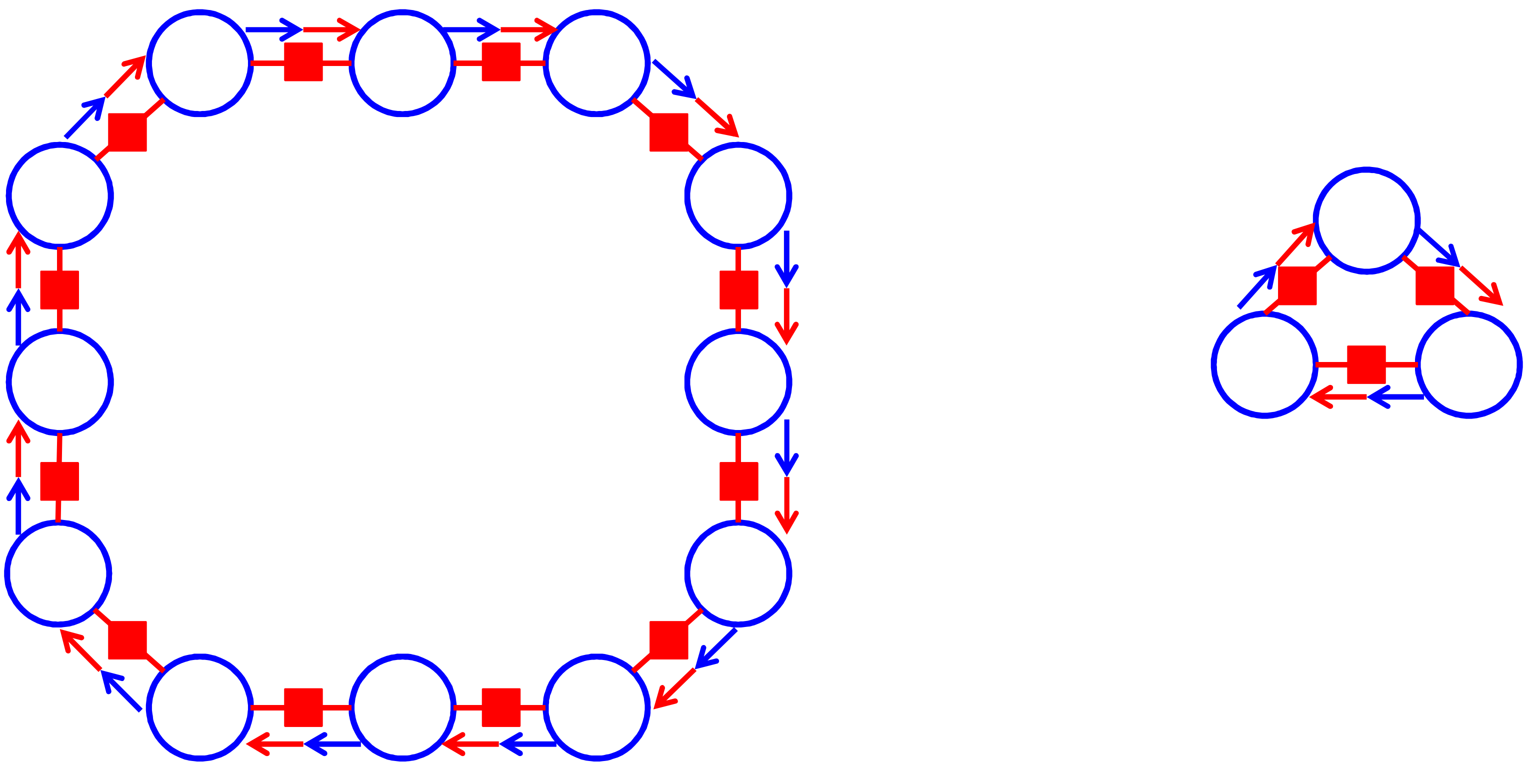


Playing Telephone



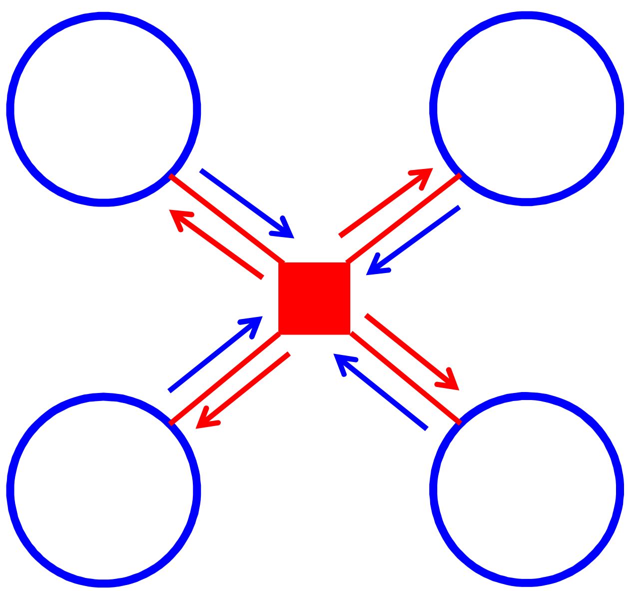
Part 5: Belief Propagation with Structured Factors

Structured Factors

- Problem:
  - Computing factor messages is exponential in arity
  - Many models we care about have high-arity factors
- Solution:
  - Take advantage of NLP tricks for efficient sums
- Examples:
  - Word Alignment (at-most-one constraints)
  - Dependency Parsing (tree constraint)

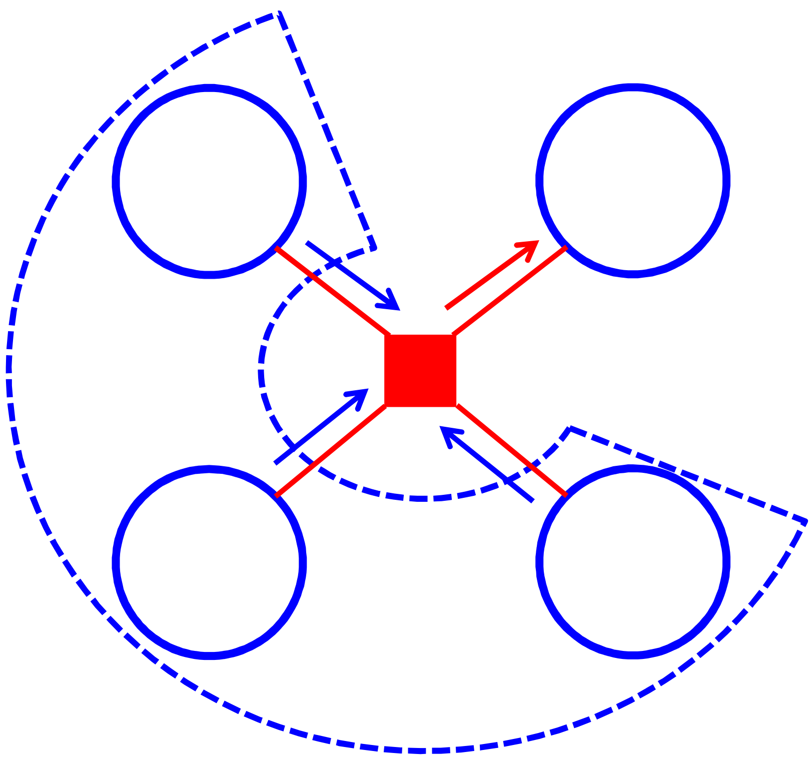
Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_c \setminus \{i\}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



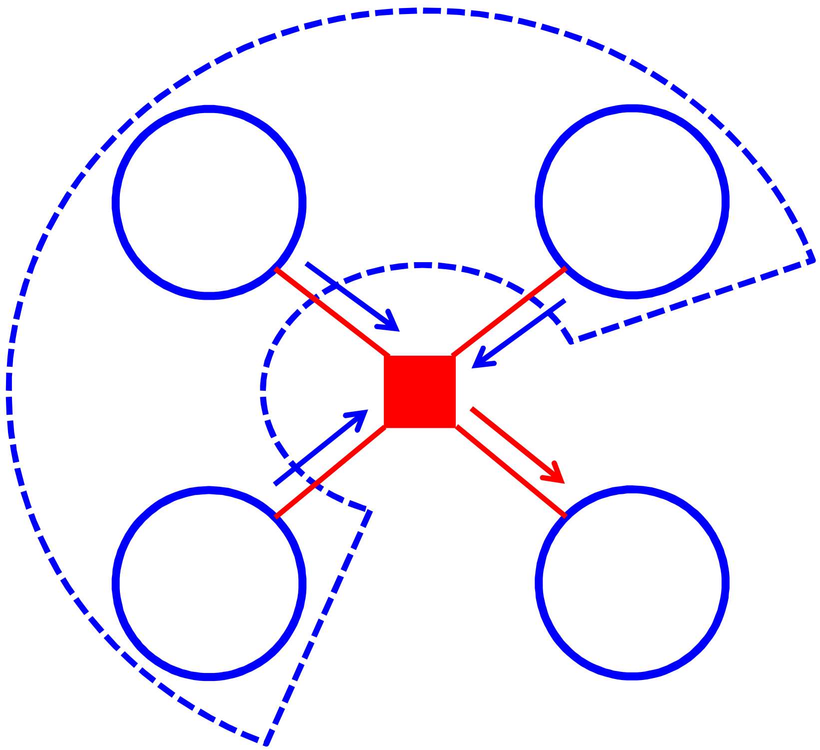
Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_c \setminus \{i\}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



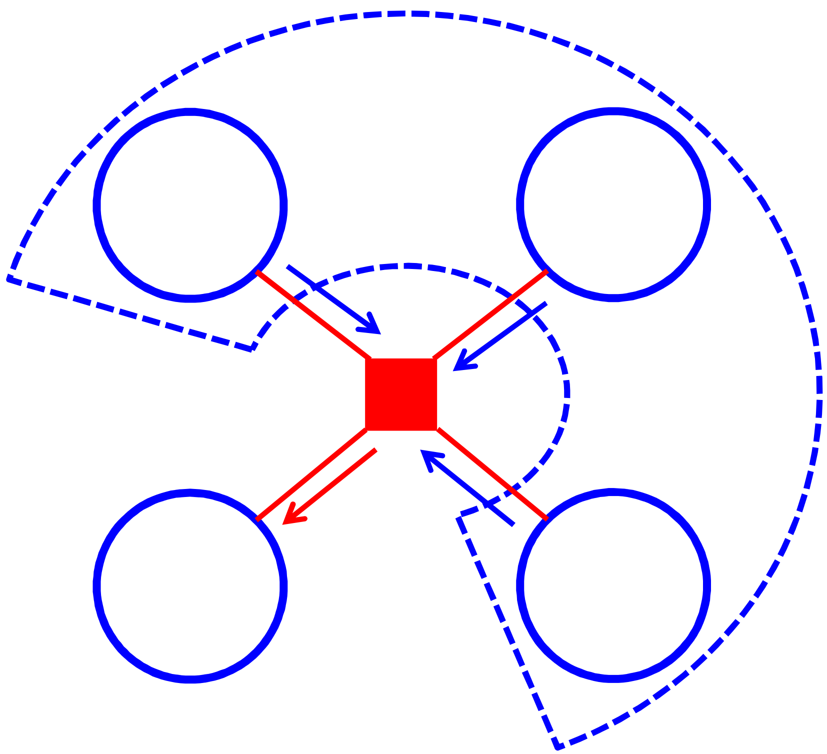
Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_{c \setminus \{i\}}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



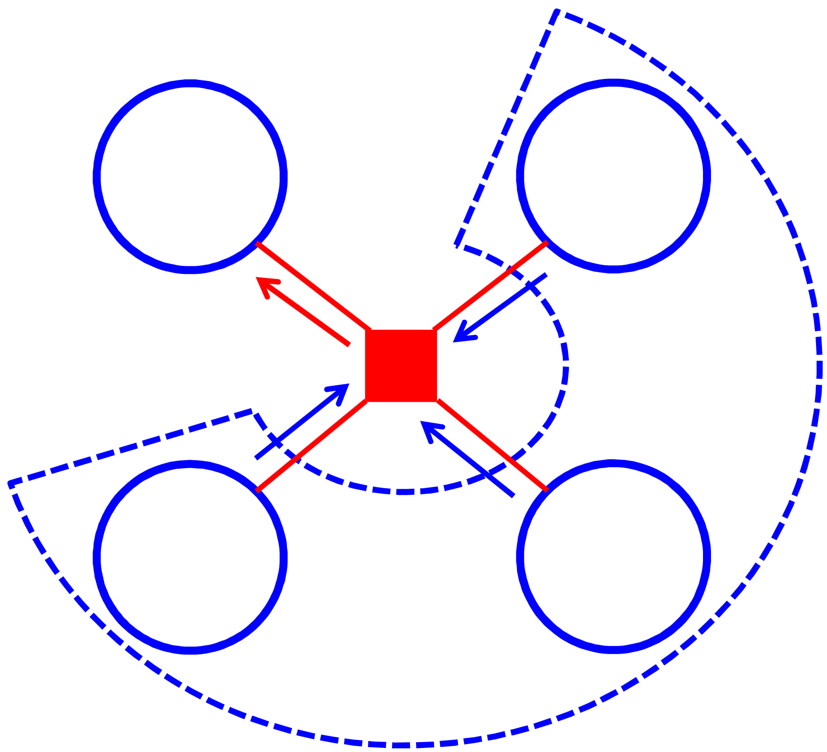
Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_c \setminus \{i\}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



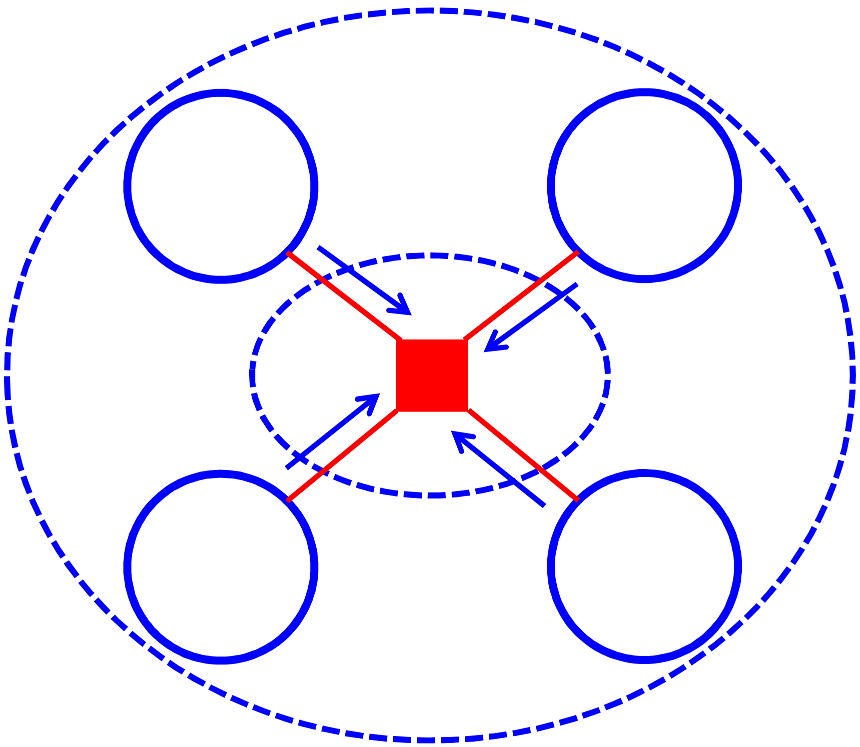
Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_{c \setminus \{i\}}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



Warm-up Exercise

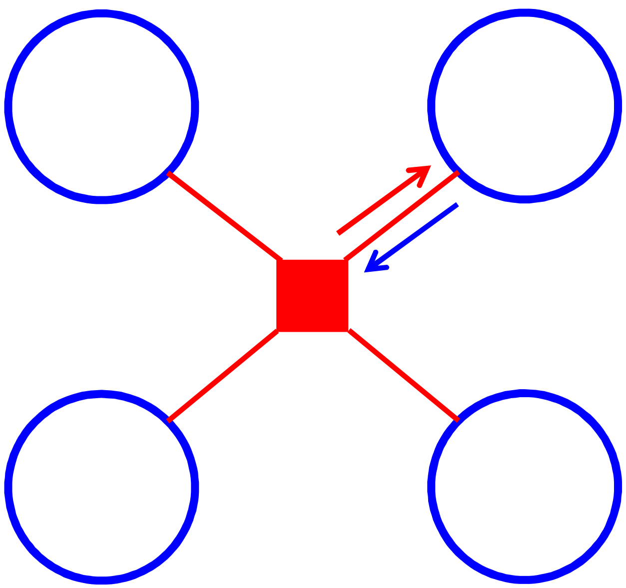
$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_{c \setminus \{i\}}} \phi_c(y_c) \prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



$$s(y_c) = \prod_{i \in c} m_{Y_i \to \phi_c}(y_i)$$

Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_{c\setminus\{i\}}} \phi_c(y_c) \prod_{i' \in c\setminus\{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})$$



$$= \frac{s(y_c)}{m_{Y_i \to \phi_c}(y_i)}$$

$$s(y_c) = \prod_{i \in c} m_{Y_i \to \phi_c}(y_i)$$

Warm-up Exercise

$$m_{\phi_c \to Y_i}(y_i) \propto \sum_{y_c \setminus \{i\}} \phi_c(y_c) \boxed{\prod_{i' \in c \setminus \{i\}} m_{Y_{i'} \to \phi_c}(y_{i'})}$$

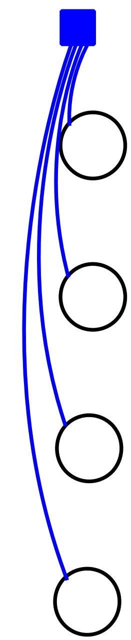
$$= \frac{s(y_c)}{m_{Y_i \to \phi_c}(y_i)}$$

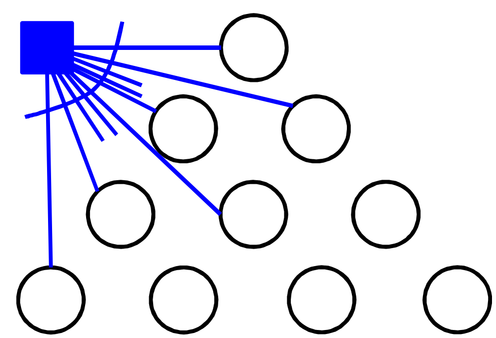
- Benefits:
  - Cleans up notation
  - Saves time multiplying
  - Enables efficient summing

$$\boxed{s(y_c)} = \prod_{i \in c} m_{Y_i \to \phi_c}(y_i)$$

The Shape of Structured BP

- Isolate the combinatorial factors
- Figure out how to compute efficient sums
  - Directly exploiting sparsity
  - Dynamic programming
- Work out the bookkeeping
  - Or, use a reference!





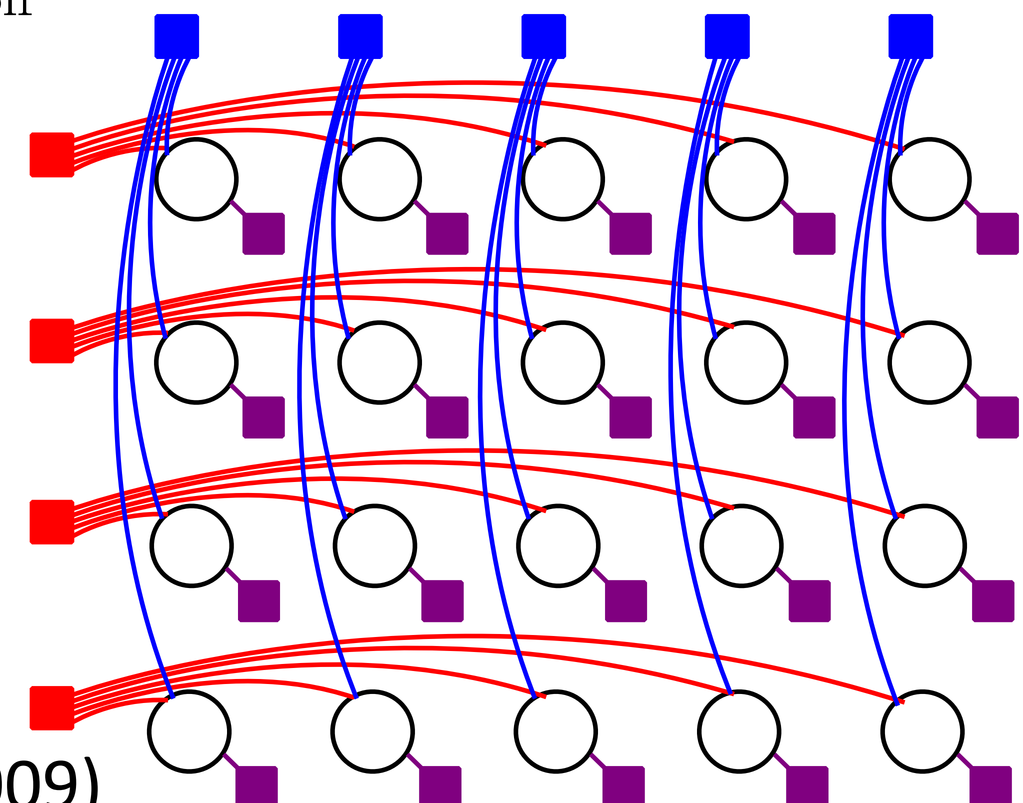
Word Alignment with BP

$$\phi(y_{ij}) = \begin{cases} \exp(w^\top f(i,j)) & y_{ij} = \text{on} \\ 1 & y_{ij} = \text{off} \end{cases}$$

$$\phi(y_{i*}) = \begin{cases} 1 & |\{j : y_{ij} = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(y_{*j}) = \begin{cases} 1 & |\{i : y_{ij} = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$y_{ij} \in \{\text{on}, \text{off}\}$$



(Cromières & Kurohashi, 2009)

(Burkett & Klein, 2012)

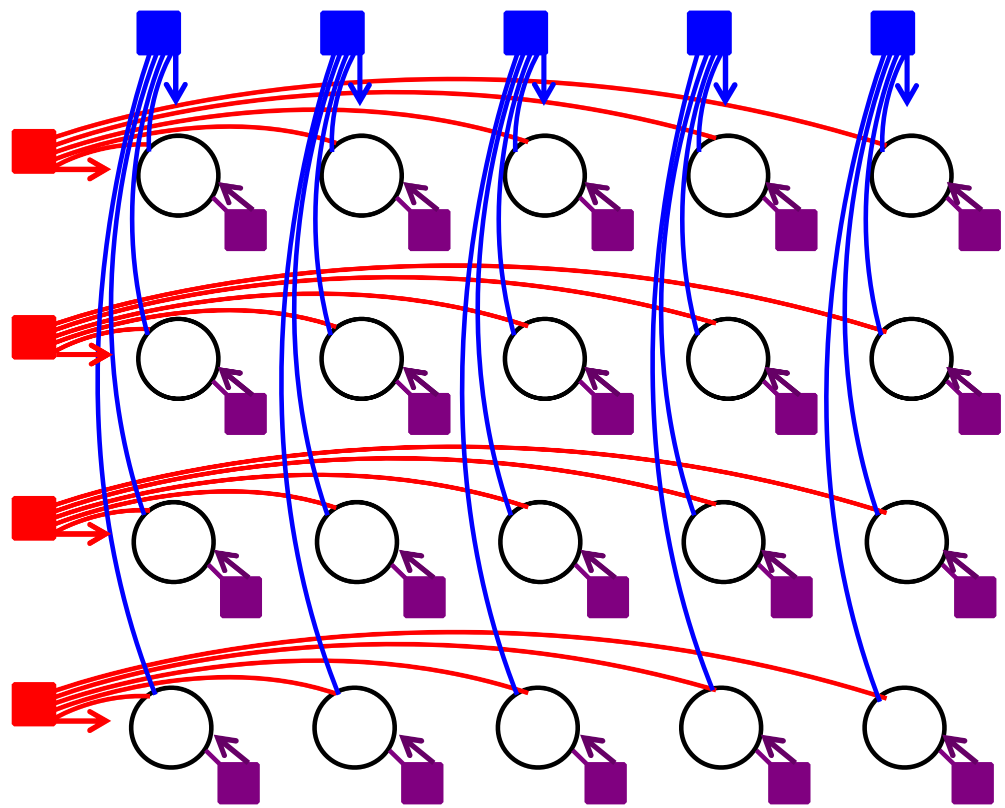
Computing Messages from Factors

Exponential in arity of factor
(have to sum over all assignments)

$\phi_{ij}(y_{ij})$ Arity 1

$\phi_i(y_{i*})$ Arity $O(n)$

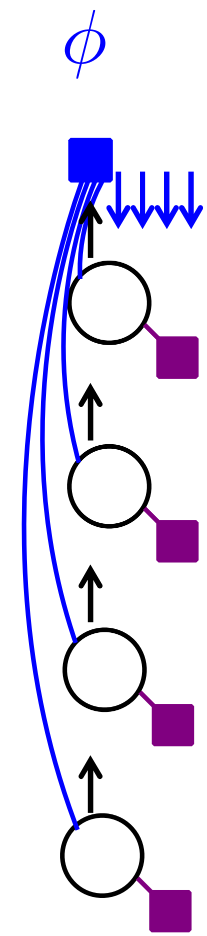
$\phi_j(y_{*j})$ Arity $O(n)$



Computing Constraint Factor Messages

- Input: $m_{Y_j \to \phi}(y_j)\ \forall j$
- Goal: $m_{\phi \to Y_j}(y_j)\ \forall j$

$$m_{\phi \to Y_j}(y_j) \propto \frac{\sum_{y:\ Y_j = y_j} \phi(y) s(y)}{m_{Y_j \to \phi}(y_j)}$$

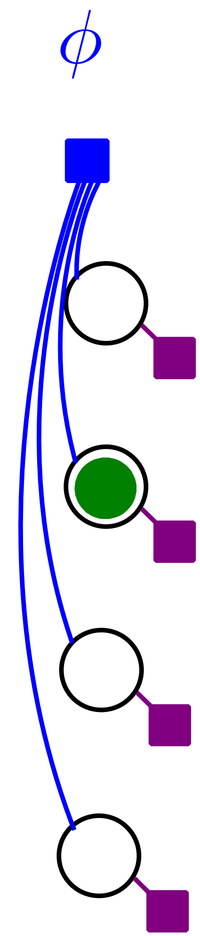


Computing Constraint Factor Messages

$y(j)$: Assignment to variables where $Y_j = \text{on}$

$$y(2) = \{Y_1 = \text{off}, \; Y_2 = \text{on}, \; Y_3 = \text{off}, \; Y_4 = \text{off}\}$$

$$\phi(y) = \begin{cases} 1 & |\{j : y_j = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

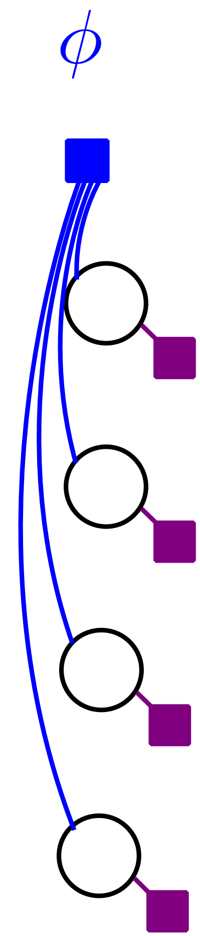


Computing Constraint Factor Messages

$y(j)$: Assignment to variables where $Y_j = \text{on}$

$y(0)$: Special case for all off

$$y(0) = \{Y_1 = \text{off}, \\ Y_2 = \text{off}, \\ Y_3 = \text{off}, \\ Y_4 = \text{off}\}$$



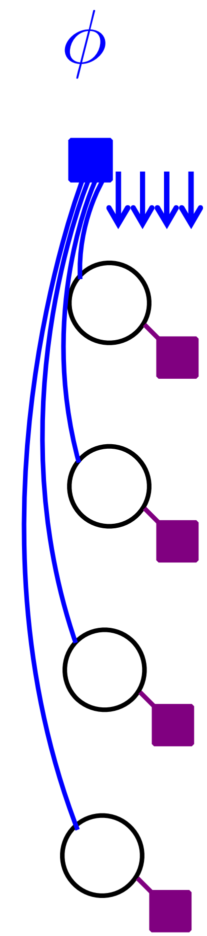
$$\phi(y) = \begin{cases} 1 & |\{j : y_j = \text{on}\}| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Computing Constraint Factor Messages

- Input: $m_{Y_j \to \phi}(y_j)\ \forall j$
- Goal: $m_{\phi \to Y_j}(y_j)\ \forall j$

$$m_{\phi \to Y_j}(y_j) \propto \frac{\sum_{y:\ Y_j = y_j} \phi(y) s(y)}{m_{Y_j \to \phi}(y_j)}$$

Only need to consider
$y(j')$ for $0 \le j' \le n$

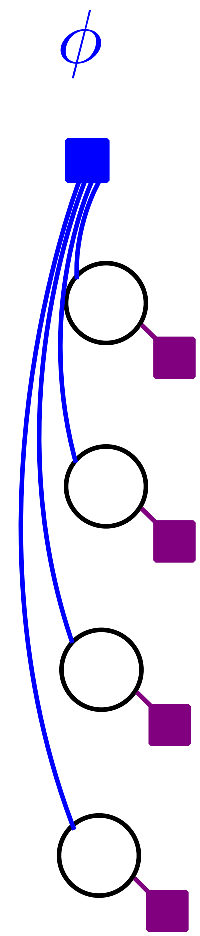


Computing Constraint Factor Messages

$$s(y) = \prod_j m_{Y_i \to \phi}(y_j)$$

$$\begin{aligned} s(y(0)) = \; & m_{Y_1 \to \phi}(\text{off}) \cdot \\ & m_{Y_2 \to \phi}(\text{off}) \cdot \\ & m_{Y_3 \to \phi}(\text{off}) \cdot \\ & m_{Y_4 \to \phi}(\text{off}) \end{aligned}$$

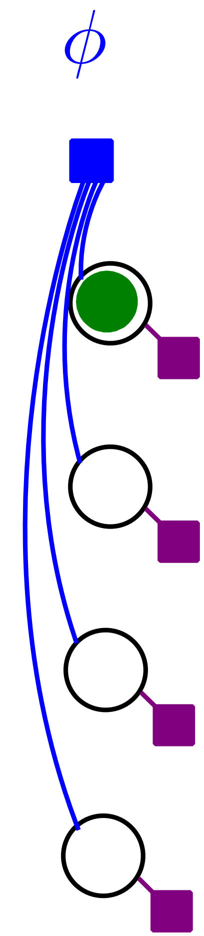
$$s(y(0)) = \prod_{1 \le j \le n} m_{Y_j \to \phi}(\text{off})$$



Computing Constraint Factor Messages

$$s(y) = \prod_j m_{Y_i \to \phi}(y_j)$$

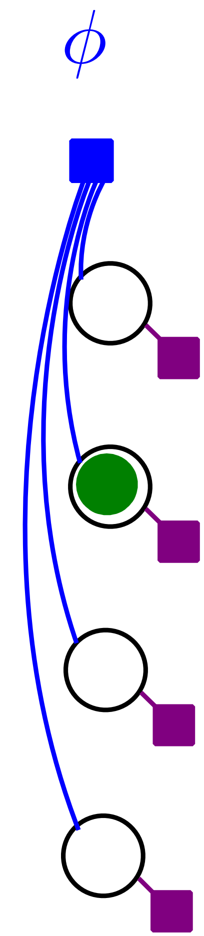
$$\begin{aligned} s(y(1)) = \; & m_{Y_1 \to \phi}(\text{on}) \cdot \\ & m_{Y_2 \to \phi}(\text{off}) \cdot \\ & m_{Y_3 \to \phi}(\text{off}) \cdot \\ & m_{Y_4 \to \phi}(\text{off}) \end{aligned}$$



Computing Constraint Factor Messages

$$s(y) = \prod_j m_{Y_i \to \phi}(y_j)$$

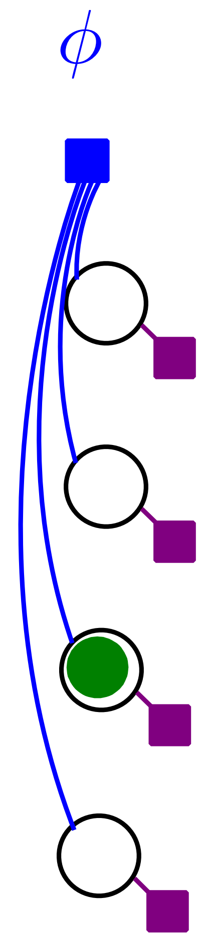
$$\begin{aligned} s(y(2)) = \; & m_{Y_1 \to \phi}(\text{off}) \cdot \\ & m_{Y_2 \to \phi}(\text{on}) \cdot \\ & m_{Y_3 \to \phi}(\text{off}) \cdot \\ & m_{Y_4 \to \phi}(\text{off}) \end{aligned}$$



Computing Constraint Factor Messages

$$s(y) = \prod_j m_{Y_i \to \phi}(y_j)$$

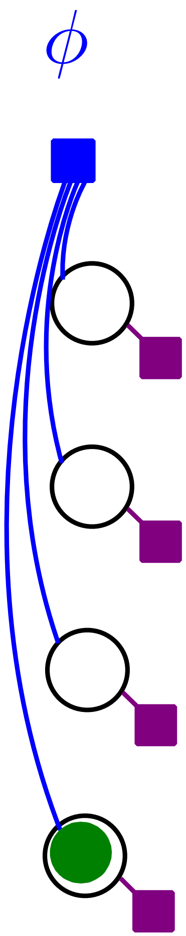
$$\begin{aligned} s(y(3)) = \; & m_{Y_1 \to \phi}(\text{off}) \cdot \\ & m_{Y_2 \to \phi}(\text{off}) \cdot \\ & m_{Y_3 \to \phi}(\text{on}) \cdot \\ & m_{Y_4 \to \phi}(\text{off}) \end{aligned}$$



Computing Constraint Factor Messages

$$s(y) = \prod_j m_{Y_i \to \phi}(y_j)$$

$$\begin{aligned} s(y(4)) = \; & m_{Y_1 \to \phi}(\text{off}) \cdot \\ & m_{Y_2 \to \phi}(\text{off}) \cdot \\ & m_{Y_3 \to \phi}(\text{off}) \cdot \\ & m_{Y_4 \to \phi}(\text{on}) \end{aligned}$$



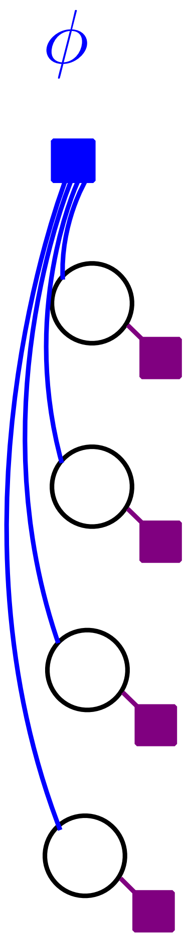
Computing Constraint Factor Messages

$$s(y) = \prod_{j} m_{Y_i \to \color{blue}{\phi}}(y_j)$$

$$s(y(0)) = \prod_{1 \leq j \leq n} m_{Y_j \to \color{blue}{\phi}}(\mathrm{off})$$

$\forall j > 0:$

$$s(y(j)) = s(y(0)) \frac{m_{Y_j \to \color{blue}{\phi}}(\mathrm{on})}{m_{Y_j \to \color{blue}{\phi}}(\mathrm{off})}$$

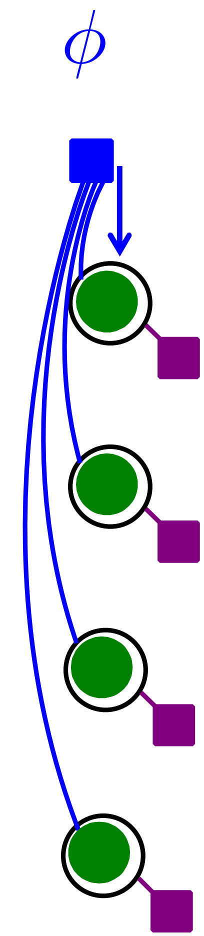


Computing Constraint Factor Messages

$$m_{\phi \rightarrow Y_1}(\text{on}) \propto \frac{s(y(1))}{m_{Y_1 \rightarrow \phi}(\text{on})}$$

$$m_{\phi \rightarrow Y_1}(\text{off}) \propto \frac{s(*) - s(y(1))}{m_{Y_1 \rightarrow \phi}(\text{off})}$$

$$s(*) = \sum_{0 \leq j \leq n} s(y(j))$$



Computing Constraint Factor Messages

1. Precompute: $s(y(0)) = \prod_{1 \leq j \leq n} m_{Y_j \to \phi}(\text{off})$
$O(n)$

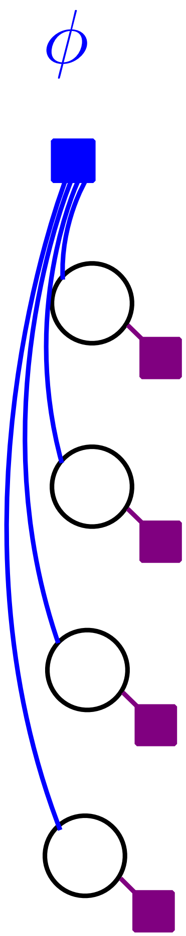
2. $\forall j > 0 : \; s(y(j)) = s(y(0)) \dfrac{m_{Y_j \to \phi}(\text{on})}{m_{Y_j \to \phi}(\text{off})}$
$O(n)$

3. Partition: $s(*) = \sum_{0 \leq j \leq n} s(y(j))$
$O(n)$

4. Messages:
$O(n)$

$$m_{\phi \to Y_j}(\text{on}) \propto \frac{s(y(j))}{m_{Y_j \to \phi}(\text{on})}$$

$$m_{\phi \to Y_j}(\text{off}) \propto \frac{s(*) - s(y(j))}{m_{Y_j \to \phi}(\text{off})}$$



Using BP Marginals

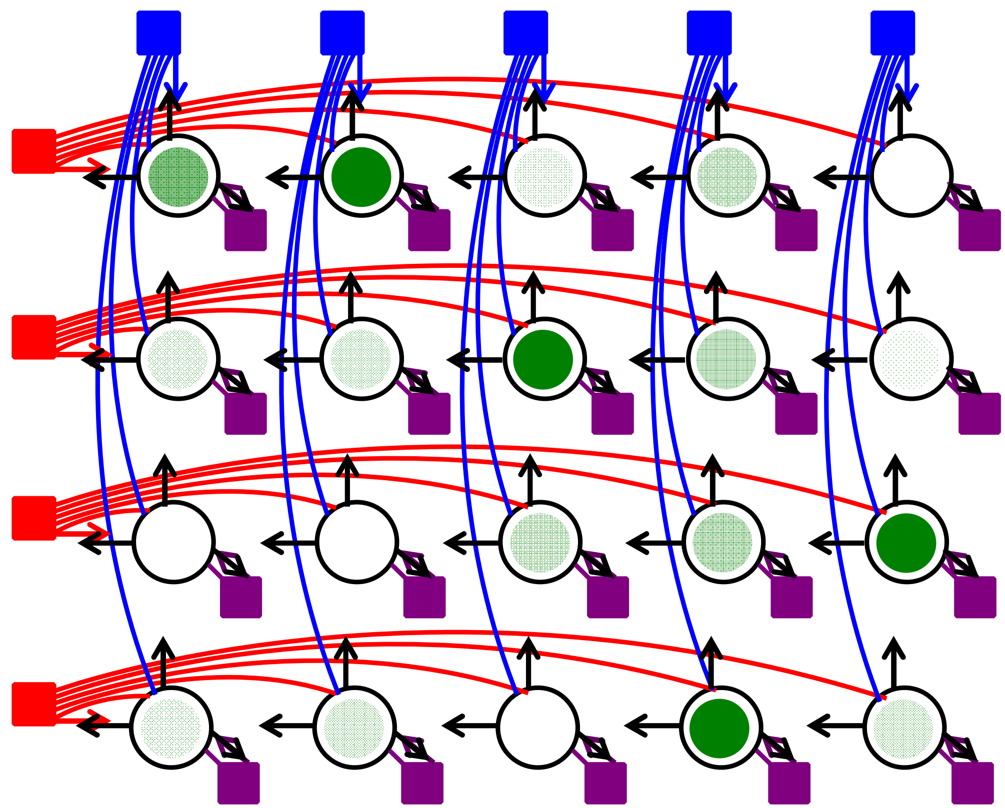
$$P(y_{ij}|x) \approx b_{Y_{ij}}(y_{ij})$$

Expected Feature Counts:

$$\mathbb{E}f(i,j) \approx b_{Y_{ij}}(\text{on}) f(i,j)$$

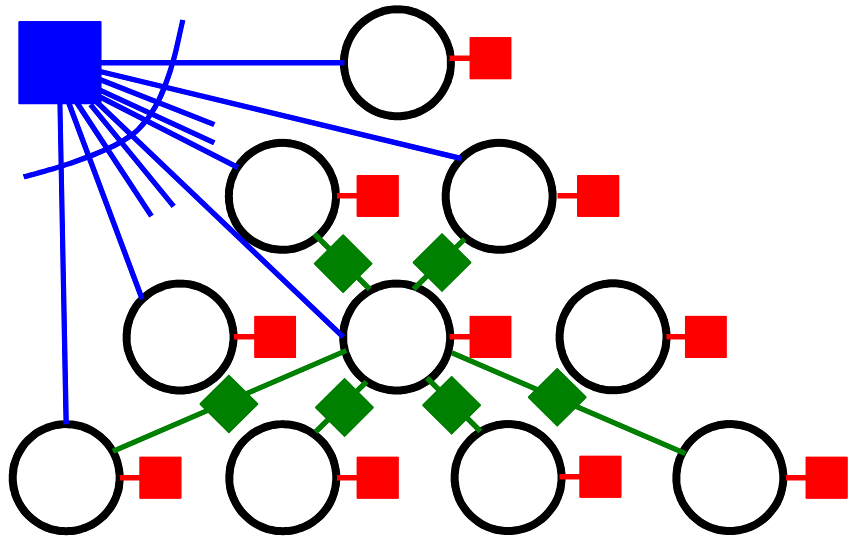
Marginal Decoding:

$$\hat{y}_{ij} = \begin{cases} \text{on} & b_{Y_{ij}}(\text{on}) \geq \tau \\ \text{off} & \text{otherwise} \end{cases}$$



Dependency Parsing with BP

$y_{ij} \in \{\text{left, right, off}\}$



$$\phi(y) = \begin{cases} 1 & y \text{ forms a tree} \\ 0 & \text{otherwise} \end{cases}$$

$$\phi(y_{ij}) = \begin{cases} \exp(w^\top f(i,j)) & y_{ij} = \text{left} \\ \exp(w^\top f(j,i)) & y_{ij} = \text{right} \\ 1 & y_{ij} = \text{off} \end{cases}$$

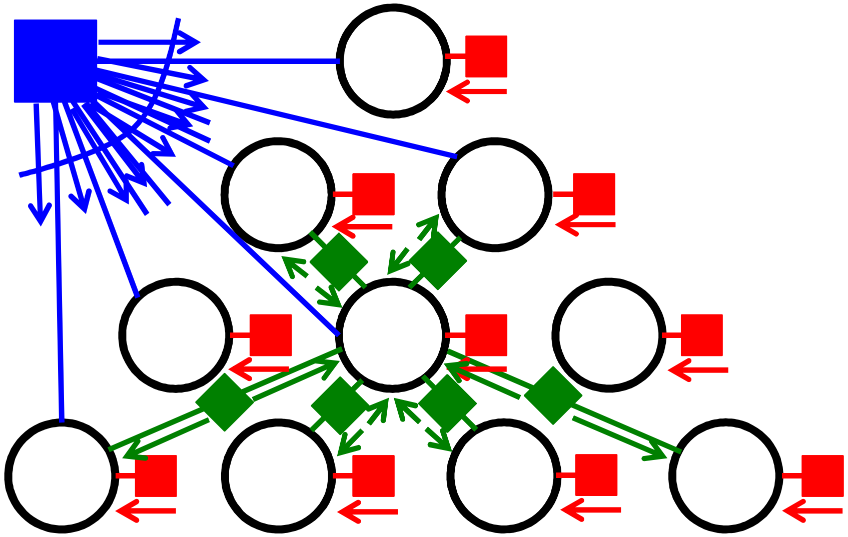
$\phi(y_{ij}, y_{jk})$

(Smith & Eisner, 2008)

(Martins et al., 2010)

Berkeley NLP

Dependency Parsing with BP



$\phi_{ij}(y_{ij})$ Arity 1 ✓

$\phi_{ijk}(y_{ij}, y_{jk})$ Arity 2 ✓

$\phi_{\text{TREE}}(y)$ Arity $O(n^2)$ ✗

Exponential in arity of factor

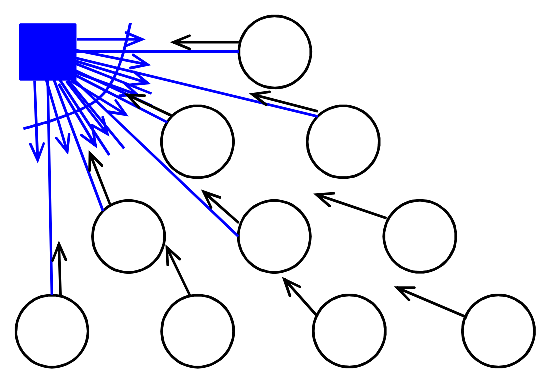
Messages from the Tree Factor

- Input: $m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})$ for all variables

- Goal: $m_{\phi_{\text{TREE}} \to Y_{ij}}(y_{ij})$ for all variables

$$m_{\phi_{\text{TREE}} \to Y_{ij}}(y_{ij}) \propto \sum \phi_{\text{TREE}}(y) s(y) \frac{1}{m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})}$$

$$\phi_{\text{TREE}}(y) = \begin{cases} 1 & y \text{ forms a tree} \\ 0 & \text{otherwise} \end{cases}$$



$$T = \{y \; : \; y \text{ forms a tree}\}$$

What Do Parsers Do?

- Initial state:
  - Value of an edge (i has parent j): $v(i,j)$
  - Value of a tree: $v(t) = \prod_{(i,j) \in t} v(i,j)$
- Run inside-outside to compute:
  - Total score for all trees: $Z = \sum_t v(t)$
  - Total score for an edge: $Z(i,j) = \sum_{t:\ (i,j) \in t} v(t)$

(Klein & Manning, 2002)

Initializing the Parser

Problem:

$$v(t) = \prod_{(i,j) \in t} v(i,j)$$

Product over edges in t: $y_{ij} = \text{left}$ or $y_{ji} = \text{right}$

$$s(y) = \prod_{ij} m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})$$

Product over ALL edges, including $y_{ij} = \text{off}$

Solution: Use odds ratios

$$v(y_{ij}) = \begin{cases} \dfrac{m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})}{m_{Y_{ij} \to \phi_{\text{TREE}}}(\text{off})} & y_{ij} \neq \text{off} \\ 1 & y_{ij} = \text{off} \end{cases}$$

$$\pi = \prod_{ij} m_{Y_{ij} \to \phi_{\text{TREE}}}(\text{off})$$

$$\pi v(t) = s(y)$$

Running the Parser

$$Z = \sum_t v(t) \qquad \pi v(t) = s(y) \qquad \pi Z = \sum_{y \in T} s(y)$$

Sums we want:

$$\pi Z(i,j) = \sum_{\substack{y \in T \\ y_{ij} = \text{left}}} s(y) \qquad \pi Z(j,i) = \sum_{\substack{y \in T \\ y_{ij} = \text{right}}} s(y)$$

$$\pi(Z - Z(i,j) - Z(j,i)) = \sum_{\substack{y \in T \\ y_{ij} = \text{off}}} s(y)$$

Computing Tree Factor Messages

1. Precompute: $\pi = \prod_{ij} m_{Y_{ij} \to \phi_{\text{TREE}}}(\text{off})$

2. Initialize: $v(i,j) = \begin{cases} \dfrac{m_{Y_{ij} \to \phi_{\text{TREE}}}(\text{left})}{m_{Y_{ij} \to \phi_{\text{TREE}}}(\text{off})} & i < j \\[2ex] \dfrac{m_{Y_{ji} \to \phi_{\text{TREE}}}(\text{right})}{m_{Y_{ji} \to \phi_{\text{TREE}}}(\text{off})} & j < i \end{cases}$

3. Run inside-outside

4. Messages:

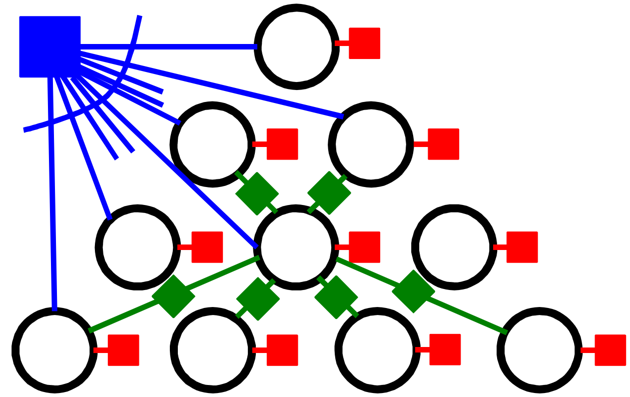
$$m_{\phi_{\text{TREE}} \to Y_{ij}}(y_{ij}) \propto \begin{cases} \dfrac{\pi Z(i,j)}{m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})} & y_{ij} = \text{left} \\[2ex] \dfrac{\pi Z(j,i)}{m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})} & y_{ij} = \text{right} \\[2ex] \dfrac{\pi (Z - Z(i,j) - Z(j,i))}{m_{Y_{ij} \to \phi_{\text{TREE}}}(y_{ij})} & y_{ij} = \text{off} \end{cases}$$

Using BP Marginals

$$P(y_{ij}|x) \approx b_{Y_{ij}}(y_{ij})$$

- Expected Feature Counts:

$$\mathbb{E}f(i,j) \approx \begin{cases} b_{Y_{ij}}(\text{left}) f(i,j) & i < j \\ b_{Y_{ji}}(\text{right}) f(i,j) & j < i \end{cases}$$



- Minimum Risk Decoding:

1. Initialize:

$$v(i,j) = \begin{cases} \dfrac{b_{Y_{ij}}(\text{left})}{b_{Y_{ij}}(\text{off})} & i < j \\[2ex] \dfrac{b_{Y_{ji}}(\text{right})}{b_{Y_{ji}}(\text{off})} & j < i \end{cases}$$

2. Run parser:

$$\hat{t} = \operatorname*{argmax}_t s(t)$$

Structured BP Summary

- Tricky part is factors whose arity grows with input size
- Simplify the problem by focusing on sums of total scores
- Exploit problem-specific structure to compute sums efficiently
- Use odds ratios to eliminate "default" values that don't appear in dynamic program sums

Belief Propagation Tips

- Don't compute unary messages multiple times
- Store variable beliefs to save time computing variable to factor messages (divide one out)
- Update the slowest messages less frequently
- You don't usually need to run to convergence; measure the speed/performance tradeoff

Part 6: Wrap-Up

Mean Field vs Belief Propagation

- When to use Mean Field:
  - Models made up of weakly interacting structures that are individually tractable
  - Joint models often have this flavor
- When to use Belief Propagation:
  - Models with intersecting factors that are tractable in isolation but interact badly
  - You often get models like this when adding non-local features to an existing tractable model

Mean Field vs Belief Propagation

- Mean Field Advantages
  - For models where it applies, the coordinate ascent procedure converges quite quickly
- Belief Propagation Advantages
  - More broadly applicable
  - More freedom to focus on factor graph design when modeling
- Advantages of Both
  - Work pretty well when the real posterior is peaked (like in NLP models!)

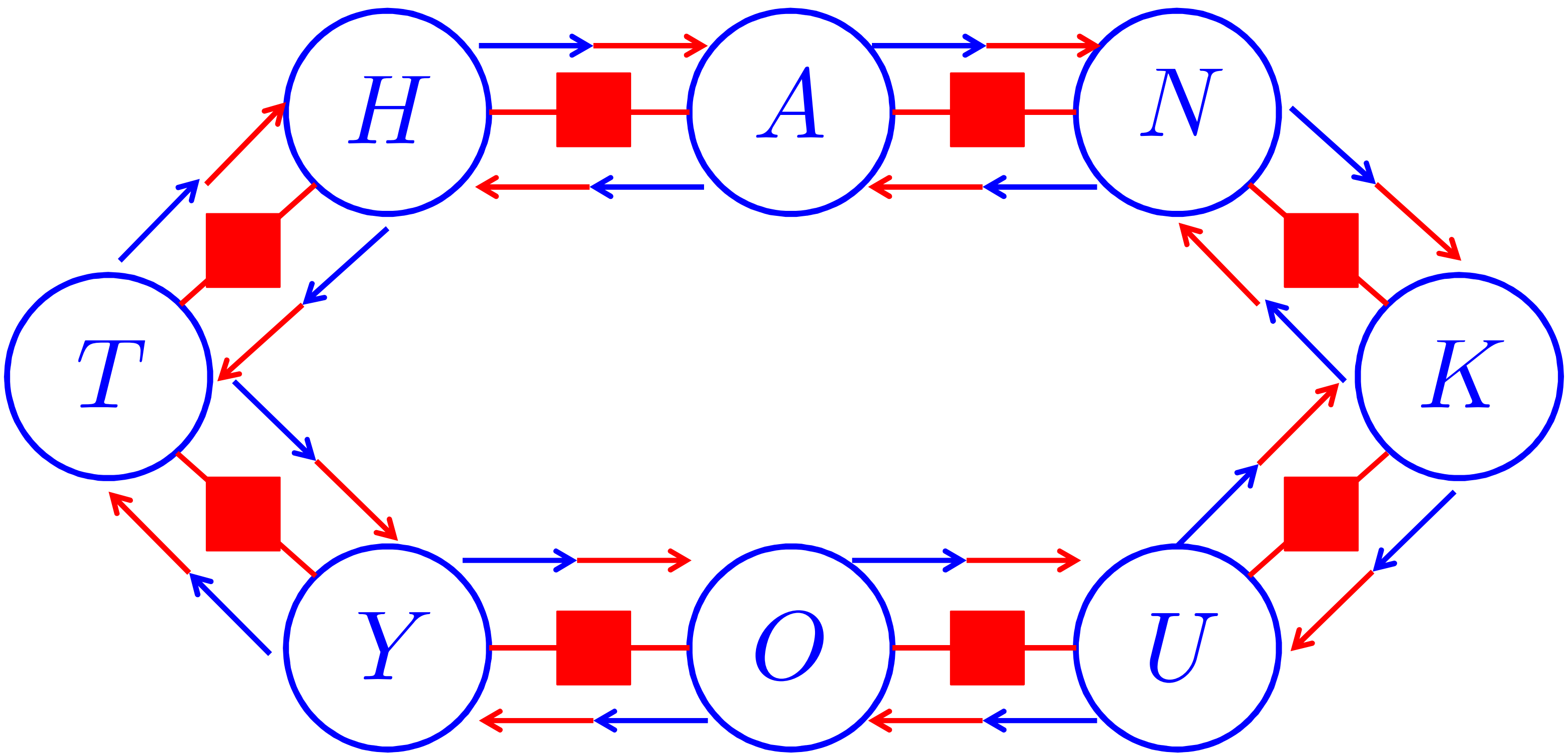
Other Variational Techniques

- Variational Bayes
  - Mean Field for models with parametric forms (e.g. Liang et al., 2007; Cohen et al., 2010)
- Expectation Propagation
  - Theoretical generalization of BP
  - Works kind of like Mean Field in practice; good for product models (e.g. Hall and Klein, 2012)
- Convex Relaxation
  - Optimize a convex approximate objective

Related Techniques

- Dual Decomposition
  - Not probabilistic, but good for finding maxes in similar models (e.g. Koo et al., 2010; DeNero & Machery, 2011)
- Search approximations
  - E.g. pruning, beam search, reranking
  - Orthogonal to approximate inference techniques (and often stackable!)

Thank You



Appendix A: Bibliography

References

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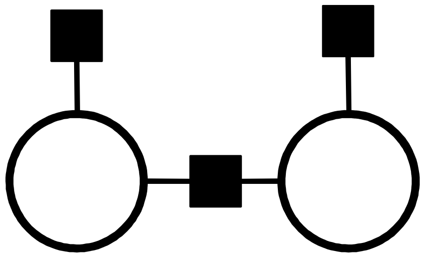
Appendix B: Mean Field Update Derivation

Berkeley NLP

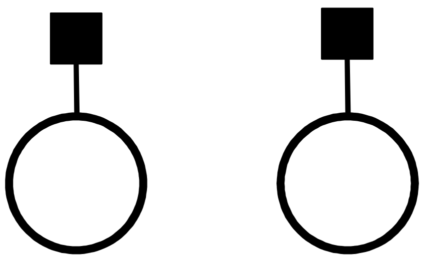
Mean Field Update Derivation

Model:



$$p(y) \propto \phi(y_1)\phi(y_2)\phi(y_1, y_2)$$

Approximate Graph:



$$q(y) = q(y_1)q(y_2)$$

Goal: $$q(y_1) = \underset{q(y_1)}{\operatorname{argmin}} KL(q||p)$$

Mean Field Update Derivation

$$KL(q||p) = \sum_{y_1, y_2} q(y_1)q(y_2) \frac{\log q(y_1)q(y_2)}{\log p(y_1, y_2)}$$

Mean Field Update Derivation

$$KL(q||p) = \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)}$$

$$= \sum_{y_1,y_2} q(y_1)q(y_2)$$

Mean Field Update Derivation

$$KL(q||p) = \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\boxed{\log q(y_1)q(y_2)}}{\log p(y_1,y_2)}$$

$$= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2)\right.$$

Mean Field Update Derivation

$$
\begin{aligned}
KL(q||p) &= \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\boxed{\log p(y_1,y_2)}} \\
&= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log \phi(y_1) - \log \phi(y_2) - \log \phi(y_1,y_2) + \log Z_x\right)
\end{aligned}
$$

Mean Field Update Derivation

$$KL(q||p) = \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)}$$

$$= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log \phi(y_1) - \log \phi(y_2) - \log \phi(y_1,y_2) + \log Z_x\right)$$

$$= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log \phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log \phi(y_1,y_2)\right) +$$

$$\left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log \phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right)$$

Mean Field Update Derivation

$$
\begin{aligned}
KL(q||p) &= \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)} \\
&= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log\phi(y_1) - \log\phi(y_2) - \log\phi(y_1,y_2) + \log Z_x\right) \\
&= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right) \\
&= \left(\sum_{y_1} q(y_1)\log q(y_1)\right) - \left(\sum_{y_1} q(y_1)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_2} q(y_2)\log q(y_2)\right) - \left(\sum_{y_2} q(y_2)\log\phi(y_2)\right) + \log Z_x
\end{aligned}
$$

Mean Field Update Derivation

$$
\begin{aligned}
KL(q||p) &= \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)} \\
&= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log\phi(y_1) - \log\phi(y_2) - \log\phi(y_1,y_2) + \log Z_x\right) \\
&= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right) \\
&= \left(\sum_{y_1} q(y_1)\log q(y_1)\right) - \left(\sum_{y_1} q(y_1)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_2} q(y_2)\log q(y_2)\right) - \left(\sum_{y_2} q(y_2)\log\phi(y_2)\right) + \log Z_x
\end{aligned}
$$

$$
\frac{\partial KL(q||p)}{\partial q(y_1)} =
$$

Mean Field Update Derivation

$$
\begin{aligned}
KL(q||p) &= \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)} \\
&= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log\phi(y_1) - \log\phi(y_2) - \log\phi(y_1,y_2) + \log Z_x\right) \\
&= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right) \\
&= \boxed{\left(\sum_{y_1} q(y_1)\log q(y_1)\right)} - \left(\sum_{y_1} q(y_1)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_2} q(y_2)\log q(y_2)\right) - \left(\sum_{y_2} q(y_2)\log\phi(y_2)\right) + \log Z_x
\end{aligned}
$$

$$
\frac{\partial KL(q||p)}{\partial q(y_1)} = (\log q(y_1) + 1)
$$

Mean Field Update Derivation

$$
\begin{aligned}
KL(q||p) &= \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)} \\
&= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log\phi(y_1) - \log\phi(y_2) - \log\phi(y_1,y_2) + \log Z_x\right) \\
&= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right) \\
&= \left(\sum_{y_1} q(y_1)\log q(y_1)\right) - \boxed{\left(\sum_{y_1} q(y_1)\log\phi(y_1)\right)} - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) + \\
&\quad \left(\sum_{y_2} q(y_2)\log q(y_2)\right) - \left(\sum_{y_2} q(y_2)\log\phi(y_2)\right) + \log Z_x
\end{aligned}
$$

$$
\frac{\partial KL(q||p)}{\partial q(y_1)} = (\log q(y_1) + 1) - \log\phi(y_1)
$$

Mean Field Update Derivation

$$KL(q||p) = \sum_{y_1,y_2} q(y_1)q(y_2)\frac{\log q(y_1)q(y_2)}{\log p(y_1,y_2)}$$

$$= \sum_{y_1,y_2} q(y_1)q(y_2)\left(\log q(y_1) + \log q(y_2) - \log\phi(y_1) - \log\phi(y_2) - \log\phi(y_1,y_2) + \log Z_x\right)$$

$$= \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right) +$$

$$\left(\sum_{y_1,y_2} q(y_1)q(y_2)\log q(y_2)\right) - \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_2)\right) + \left(\sum_{y_1,y_2} q(y_1)q(y_2)\log Z_x\right)$$

$$= \left(\sum_{y_1} q(y_1)\log q(y_1)\right) - \left(\sum_{y_1} q(y_1)\log\phi(y_1)\right) - \boxed{\left(\sum_{y_1,y_2} q(y_1)q(y_2)\log\phi(y_1,y_2)\right)} +$$

$$\cancel{\left(\sum_{y_2} q(y_2)\log q(y_2)\right) - \left(\sum_{y_2} q(y_2)\log\phi(y_2)\right) + \log Z_x}$$

$$\frac{\partial KL(q||p)}{\partial q(y_1)} = (\log q(y_1) + 1) - \log\phi(y_1) - \sum_{y_2} q(y_2)\log\phi(y1,y2)$$

Mean Field Update Derivation

$$0 = (\log q(y_1) + 1) - \log \phi(y_1) - \sum_{y_2} q(y_2) \log \phi(y_1, y_2)$$

Mean Field Update Derivation

$$0 = (\log q(y_1) + 1) - \log \phi(y_1) - \sum_{y_2} q(y_2) \log \phi(y_1, y_2)$$

$$\log q(y_1) = \log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1$$

Mean Field Update Derivation

$$0 = (\log q(y_1) + 1) - \log \phi(y_1) - \sum_{y_2} q(y_2) \log \phi(y_1, y_2)$$

$$\log q(y_1) = \log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1$$

$$q(y_1) = \exp\left(\log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1\right)$$

Mean Field Update Derivation

$$0 = (\log q(y_1) + 1) - \log \phi(y_1) - \sum_{y_2} q(y_2) \log \phi(y_1, y_2)$$

$$\log q(y_1) = \log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1$$

$$q(y_1) \propto \exp\left(\log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1 \right)$$

Mean Field Update Derivation

$$0 = (\log q(y_1) + 1) - \log \phi(y_1) - \sum_{y_2} q(y_2) \log \phi(y_1, y_2)$$

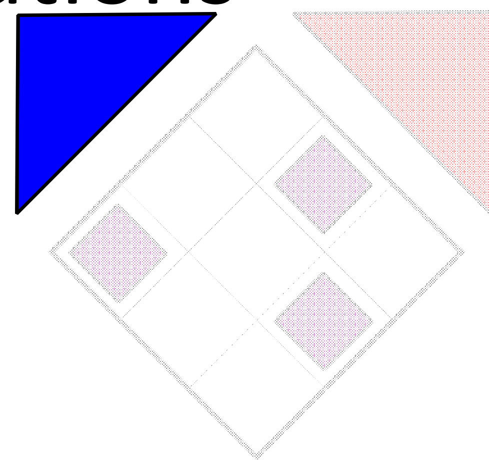
$$\log q(y_1) = \log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) - 1$$

$$q(y_1) \propto \exp \left(\log \phi(y_1) + \sum_{y_2} q(y_2) \log \phi(y_1, y_2) \right)$$

$$q(y_i) \propto \exp \left(\sum_{c:i \in c} \mathbb{E}_{q_{-Y_i}} \log \phi_c(y_c) \right)$$

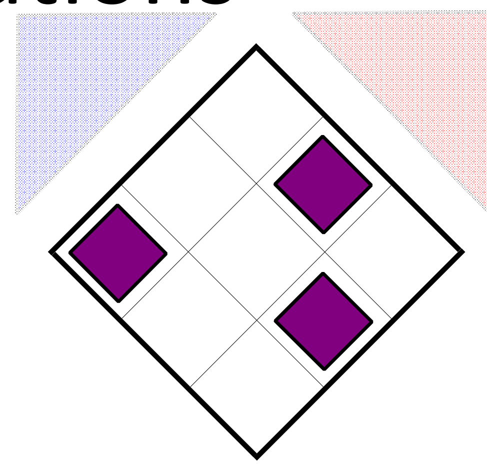
Appendix C: Joint Parsing and Alignment Component Distributions

Joint Parsing and Alignment Component Distributions



$$q(t) \propto \exp\left(\sum_{n_i X_j \in t} w^\top f_t(n) + \sum_{n_i X_j \in t} \sum_{{}_s X'_t} q(b_{ij,st})\, q(n'_{{}_s X'_t})\, w^\top f_{tat'}(n, b, n') \right)$$

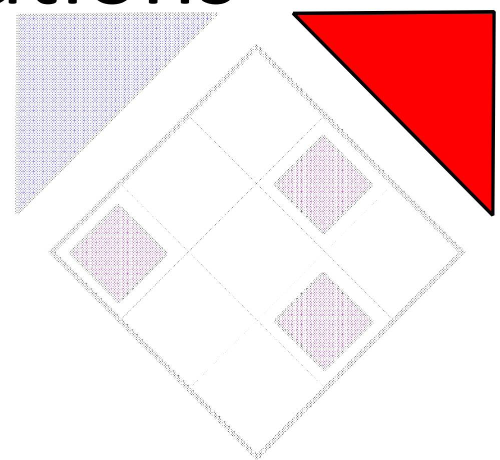
Joint Parsing and Alignment Component Distributions



$$q(a) \propto \exp\left(\sum_{b_{ij,st} \in a} w^\top f_a(b) + \right.$$

$$\left. \sum_{b_{ij,st} \in a} \sum_{X,X'} q(n_{i X_j}) q(n'_{s X'_t}) w^\top f_{tat'}(n, b, n') \right)$$

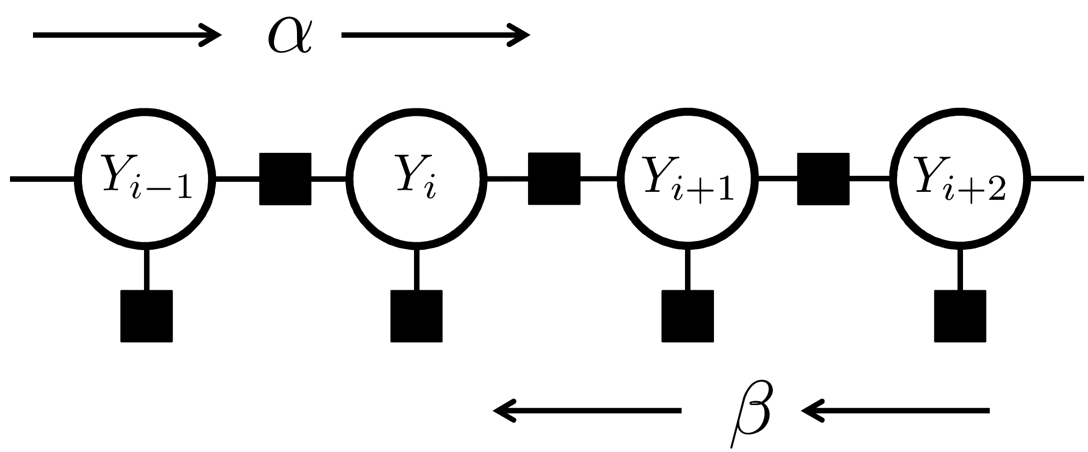
Joint Parsing and Alignment Component Distributions



$$q(t') \propto \exp\left(\sum_{n'_{{}_sX'_t} \in t'} w^\top f_{t'}(n') + \sum_{n'_{{}_sX'_t} \in t'} \sum_{{}_iX_j} q(n_{{}_iX_j})\, q(b_{ij,st})\, w^\top f_{tat'}(n, b, n')\right)$$

Appendix D: Forward-Backward as Belief Propagation

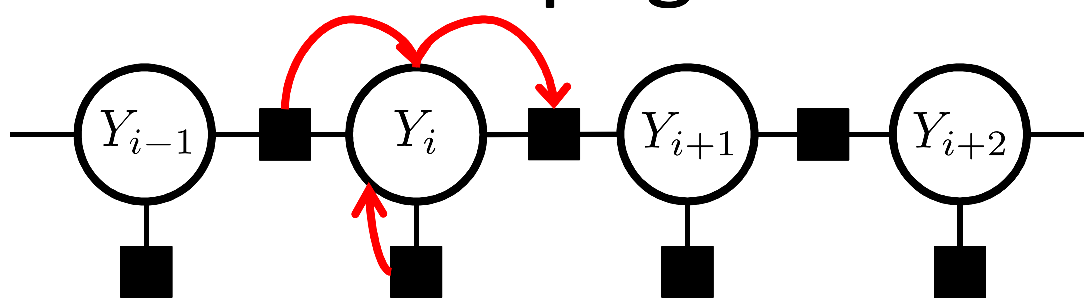
Forward-Backward as Belief Propagation



$$\alpha_i(y_i) = \phi_i(y_i) \sum_{y_{i-1}} \alpha_{i-1}(y_{i-1}) \phi_{i-1,i}(y_{i-1}, y_i)$$

$$\beta_i(y_i) = \sum_{y_{i+1}} \beta_{i+1}(y_{i+1}) \phi_{i,i+1}(y_i, y_{i+1}) \phi_{i+1}(y_{i+1})$$

Forward-Backward as Belief Propagation



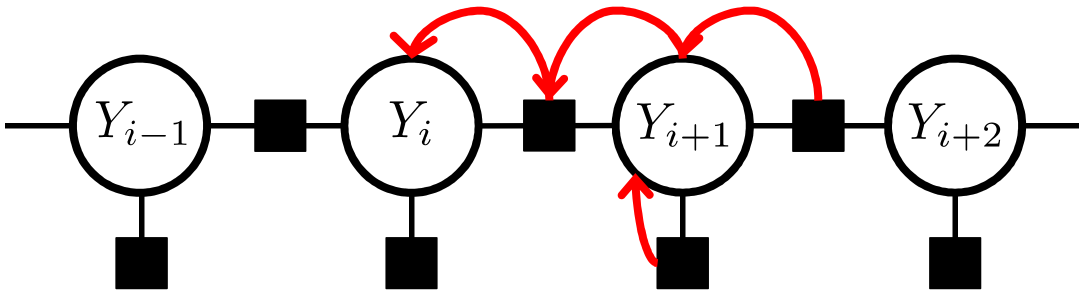
$$\alpha_i(y_i) = \phi_i(y_i) \sum_{y_{i-1}} \alpha_{i-1}(y_{i-1}) \phi_{i-1,i}(y_{i-1}, y_i)$$

$$= m_{\phi_i \to Y_i}(y_i) \; m_{\phi_{i-1,i} \to Y_i}(y_i)$$

$$m_{\phi_i \to Y_i}(y_i) = \phi_i(y_i) \qquad\qquad m_{Y_i \to \phi_{i,i+1}}(y_i) = \alpha_i(y_i)$$

$$m_{\phi_{i-1,i} \to Y_i}(y_i) = \sum_{y_{i-1}} \alpha_{i-1}(y_{i-1}) \phi_{i-1,i}(y_{i-1}, y_i)$$

Forward-Backward as Belief Propagation



$$\beta_i(y_i) = \sum_{y_{i+1}} \beta_{i+1}(y_{i+1})\phi_{i,i+1}(y_i, y_{i+1})\phi_{i+1}(y_{i+1})$$

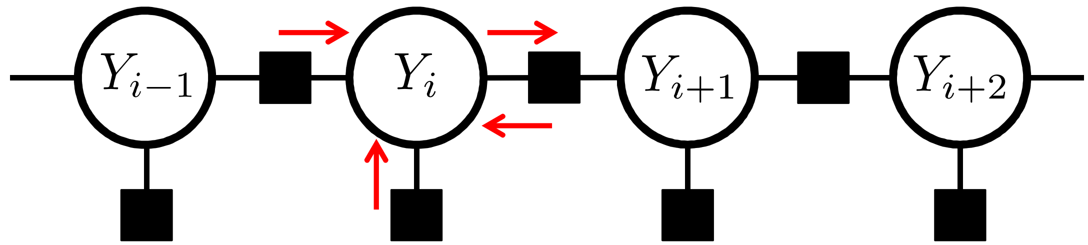
$$= m_{\phi_{i,i+1} \to Y_i}(y_i)$$

$$= \sum_{y_{i+1}} m_{Y_{i+1} \to \phi_{i,i+1}}(y_{i+1})\phi_{i,i+i}(y_i, y_{i+1})$$

$$m_{Y_{i+1} \to \phi_{i,i+1}}(y_{i+1}) = m_{\phi_{i+1} \to Y_{i+1}}(y_{i+1}) \; *$$

$$m_{\phi_{i+1,i+2} \to Y_{i+1}}(y_{i+1})$$

Forward-Backward Marginal Beliefs



$$P(y_i|x) \propto \alpha_i(y_i)\beta_i(y_i)$$

$$= m_{Y_i \to \phi_{i,i+1}}(y_i)\; m_{\phi_{i,i+1} \to Y_i}(y_i)$$

$$= m_{\phi_{i-1,i} \to Y_i}(y_i)\; m_{\phi_i \to Y_i}(y_i)\; m_{\phi_{i,i+1} \to Y_i}(y_i)$$